CHARACTERIZATION OF ELASTIC SOLIDS
USING
FINITE ELEMENT METHODS

Brian Alfred Edwards

NAVAL POSTGRADUATE SCHOOL
Monterey, California

THESIS

CHARACTERIZATION OF ELASTIC SOLIDS
USING
FINITE ELEMENT METHODS

by

Brian Alfred Edwards

Thesis Advisor: G. Cantin

December 1972

Characterization of Elastic Solids
Using
Finite Element Methods

by

Brian Alfred Edwards
Lieutenant, United States Navy
B. A., Valparaiso University, 1967
B. S., Naval Postgraduate School, 1972

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
December 1972

ABSTRACT

Finite element methods are applied to the problem of characterizing linear, anisotropic elastic solids. The conventional finite element displacement formulation is used to simulate an elastic material in plane stress. An inverted finite element formulation is then applied, and the characterizing six material constants are calculated as numerical results.

A possible test device for the experimental characterization of anisotropic solids is postulated, the precision of displacement measurements to be required for such a device being determined by random perturbation analysis. Numerical constants accurate to within three percent are predicted if a precision of one part in eight hundred (1/800) can be measured. Numerical constants accurate to within one percent are predicted if a precision of one part in eight thousand (1/8000) can be measured in the test device.

TABLE OF CONTENTS

ACKNOWLEDGEMENT

The author would like to express to Dr. Gilles Cantin, Professor of Mechanical Engineering, Naval Postgraduate School, his most ernest appreciation for the direction, guidance, and support which made this study possible. Special appreciation is gratefully tended for Dr. Cantin's friendship and encouragement.

In addition, the author extends his appreciation to Miss Kristina Bertwell and Messrs. Edwin V. Donnellan and Parker Thomas of the W. R. Church Computer Center, Naval Postgraduate School, for their professionalism and good humor.

I. INTRODUCTION

Jaensson and Sundström [11] used stress and strain approximations, developed by a finite element analysis of the microstructure of a presumed isotropic WC-Co alloy, as source data from which Young's modulus and Poisson's ratio were calculated as the characterizing constants for the alloy. A more direct application of the finite element method to the characterization of elastic solids is possible and can be applied to a wider range of materials. When finite element characterization is applied to isotropic solids an immediate advantage is secured in that Young's modulus and Poisson's ratio are obtained simultaneously. When the finite element characterization technique is applied to a general linear, anisotropic elastic material an overwhelming advantage is obtained in that no other method provides an experimental determination of all six anisotropic material constants.

The ability to characterize general linear, anisotropic elastic materials makes the design of a test device incorporating the finite element characterizing technique highly desireable. The application of this technique to linear elastic problems has been largely ignored, although an increasing use of the technique has been made to problems in the mechanical characterization of physically and kinematically non-linear materials [6, 7, 9, 12, 13, 15, 16].

The work reported in this study concerns a general class of linear, anisotropic materials. Its goal is to identify and determine preliminary design parameters applicable to the design of a testing device to be used in the characterization of linear, anisotropic solids. The finite

element technique employed in this characterization scheme has been observed [12] to demonstrate a sensitivity to the measurements of displacement data, and it is a specific goal of this study to identify such sensitivities and the degree of measurement precision required.

The study has been divided into three parts. In the first part the analytics of the finite element characterization are examined. Rather tedious derivations result in closed-form formulations, around which the characterization technique is built. The second part of the study uses a structural analysis computer code to simulate necessary test displacement data. The third and final part of the study required the writing of a computer program incorporating the finite element analytics. Perturbations of the test displacement data and consequent computer solutions to the characterization problem permitted a determination of the measurement precisions required for the possible test device.

II. THEORETICAL BACKGROUND

A. THEORY OF ELASTICITY

All structural materials exhibit in various degrees the property of elasticity: external forces loading a structure produce deformations of the structure, and if these forces do not exceed a limiting value the deformations disappear when the forces are removed. The theory of elasticity provides mathematical relations between forces and displacements acting in a structure. The action of forces lead to the definition of the stress tensor, and the geometric deformations lead to the definition of the strain tensor. A relation between these two tensors is called a constitutive law [8, 10, 14, 19]. It is this law, in its simplest formulation, which is paramount to the characterization of linear elastic solids.

1. Generalized Linear, Anisotropic Materials in Three-dimensions

The general form of the constitutive law in the theory of elasticity, a generalized Hooke's law, is the set of functions relating stresses to strains:

$$\begin{aligned}
\sigma_x &= f_1(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \\
\sigma_y &= f_2(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \\
\sigma_z &= f_3(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \\
\tau_{xy} &= f_4(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \\
\tau_{xz} &= f_5(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) \\
\tau_{yz} &= f_6(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})
\end{aligned} \tag{1}$$

where x, y, z are the axes of the orthogonal, Cartesian coordinate system; σ_x, σ_y, σ_z are the normal stresses; τ_{xy}, τ_{xz}, τ_{yz} are the

shear stresses; ε_{xx}, ε_{yy}, ε_{zz} are the normal strains; and γ_{xy}, γ_{xz}, γ_{yz} are the shear strains.

In the case of small deformations the simplest form of functions (1) is a linear set of equations. Denoting the coefficients applicable to this linear formulation by a_{mn}:

$$\begin{aligned}
\sigma_x &= a_{11}\varepsilon_{xx} + a_{12}\varepsilon_{yy} + a_{13}\varepsilon_{zz} + a_{14}\gamma_{xy} + a_{15}\gamma_{xz} + a_{16}\gamma_{yz} \\
\sigma_y &= a_{21}\varepsilon_{xx} + a_{22}\varepsilon_{yy} + a_{23}\varepsilon_{zz} + a_{24}\gamma_{xy} + a_{25}\gamma_{xz} + a_{26}\gamma_{yz} \\
\sigma_z &= a_{31}\varepsilon_{xx} + a_{32}\varepsilon_{yy} + a_{33}\varepsilon_{zz} + a_{34}\gamma_{xy} + a_{35}\gamma_{xz} + a_{36}\gamma_{yz} \\
\tau_{xy} &= a_{41}\varepsilon_{xx} + a_{42}\varepsilon_{yy} + a_{43}\varepsilon_{zz} + a_{44}\gamma_{xy} + a_{45}\gamma_{xz} + a_{46}\gamma_{yz} \\
\tau_{xz} &= a_{51}\varepsilon_{xx} + a_{52}\varepsilon_{yy} + a_{53}\varepsilon_{zz} + a_{54}\gamma_{xy} + a_{55}\gamma_{xz} + a_{56}\gamma_{yz} \\
\tau_{yz} &= a_{61}\varepsilon_{xx} + a_{62}\varepsilon_{yy} + a_{63}\varepsilon_{zz} + a_{64}\gamma_{xy} + a_{65}\gamma_{xz} + a_{66}\gamma_{yz}
\end{aligned} \tag{1a}$$

Using matrix notation, the set of functions (1a) is written:

$$\{\sigma\} = [A]\{\varepsilon\} \tag{1b}$$

or,

$$\{\varepsilon\} = [B]\{\sigma\} \tag{1c}$$

where $\{\sigma\}$ and $\{\varepsilon\}$ are the vectors whose elements are, respectively, the stress and strain components of the stress and strain tensors at a point,

$\{\sigma\} = [\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}]^T$ and $\{\varepsilon\} = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}]^T$;

and $[A]$ and $[B]$ are symmetric, 6×6 matrices of coefficients, $[A]^{-1} = [B]$, consisting of twenty-one different constants, a_{mn}.

Restricting attention to elasticity in two-dimensions, two distinct problem types are identified: plane stress problems, requiring

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0,$$

and plane strain problems, requiring

$$\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0.$$

2. <u>Elasticity in Plane Stress</u>

In the plane stress problem the in-plane stresses are defined as:

$$\{\sigma^P\} = [\sigma_x, \sigma_y, \tau_{xy}]^T.$$

Corresponding to these in-plane stresses are in-plane strains:

$$\{\varepsilon^P\} = [\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}]^T.$$

Other strain components also exist. The out-of-plane strain components are defined as:

$$\{\varepsilon^{OP}\} = [\varepsilon_{zz}, \gamma_{xz}, \gamma_{yz}]^T,$$

where $\{\varepsilon^{OP}\}$ is the vector of these out-of-plane strains.

The constitutive equation for the plane stress problem is, in terms of equation (1b):

$$\begin{Bmatrix} \sigma^P \\ 0 \end{Bmatrix} = \begin{bmatrix} A^*_{11} & A^*_{12} \\ A^{*T}_{12} & A^*_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon^P \\ \varepsilon^{OP} \end{Bmatrix}, \tag{2}$$

where

$$A^*_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{bmatrix}; \; A^*_{12} = \begin{bmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{bmatrix}; \; A^*_{22} = \begin{bmatrix} a_{33} & a_{35} & a_{36} \\ a_{53} & a_{55} & a_{56} \\ a_{63} & a_{65} & a_{66} \end{bmatrix};$$

and the a_{mn} are the linear coefficients in equation (1b).

From equation (2):

$$\{\varepsilon^{OP}\} = -[(A^*_{22})^{-1}(A^*_{12})^T]\{\varepsilon^P\} \tag{2a}$$

and,

$$\{\sigma^P\} = [A^*_{11}]\{\varepsilon^P\} + [A^*_{12}]\{\varepsilon^{OP}\}$$

$$= [A^*_{11} - (A^*_{12})(A^*_{22})^{-1}(A^*_{12})^T]\{\varepsilon^P\} \ . \tag{2b}$$

In the plane stress problem the stress components only are in-plane: the state of strain is three-dimensional with the out-of-plane components given by equation (2a).

Alternately, equation (2b) is written:

$$\{\sigma^P\} = [A']\{\varepsilon^P\} \tag{2c}$$

where $[A']$ is a symmetric, 3 x 3 matrix of constants:

$$[A'] = [A^*_{11} - (A^*_{12})(A^*_{22})^{-1}(A^*_{12})^T],$$

the determination of which characterizes a linear, anisotropic material in plane stress.

For perfectly isotropic materials submitted to a state of plane stress:

$$\left.\begin{aligned} A'_{11} &= \frac{E}{1-\nu^2} = A'_{22} \\ A'_{12} &= \nu A'_{11} \\ A'_{13} &= A'_{23} = 0 \\ A'_{33} &= \tfrac{1}{2}(1-\nu)A'_{11} \end{aligned}\right. \tag{3}$$

where E and ν are Young's modulus and Poisson's ratio, respectively, and A'_{11}, A'_{12}, A'_{13}, A'_{22}, A'_{23}, and A'_{33} are the six elastic constants forming $[A']$.

If an isotropic material with $E = 29.50 \times 10^3$ ksi and $\nu = 0.287$ is postulated and if this material were subjected to an external load configuration equivalent to plane stress, the constitutive equation becomes:

$$\{\sigma^P\} = \begin{bmatrix} 32147.998 & 9226.476 & 0.0 \\ & 32147.998 & 0.0 \\ (\text{symmetric}) & & 11460.0 \end{bmatrix} \{\varepsilon^P\} \quad . \tag{3a}$$

3. Elasticity in Plane Strain

In the plane strain problem, the in-plane strains are defined as:

$$\{\varepsilon^P\} = [\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}]^T \ .$$

Corresponding to these in-plane strains are in-plane stresses:

$$\{\sigma^P\} = [\sigma_x, \sigma_y, \tau_{xy}]^T \ .$$

Other stress components also exist, and these out-of-plane stresses are defined as:

$$\{\sigma^{OP}\} = [\sigma_z, \tau_{xz}, \tau_{yz}]^T \ ,$$

where $\{\sigma^{OP}\}$ is the vector of out-of-plane stresses.

The constitutive equation for the plane strain problem is, in terms of equation (1c):

$$\begin{Bmatrix} \varepsilon^P \\ 0 \end{Bmatrix} = \begin{bmatrix} B^*_{11} & B^*_{12} \\ B^{*T}_{12} & B^*_{22} \end{bmatrix} \begin{Bmatrix} \sigma^P \\ \sigma^{OP} \end{Bmatrix} \tag{4}$$

where

$$B^*_{11} = \begin{bmatrix} b_{11} & b_{12} & b_{14} \\ b_{21} & b_{22} & b_{24} \\ b_{41} & b_{42} & b_{44} \end{bmatrix} ; \; B^*_{12} = \begin{bmatrix} b_{13} & b_{15} & b_{16} \\ b_{23} & b_{25} & b_{26} \\ b_{43} & b_{45} & b_{46} \end{bmatrix} ; \; B^*_{22} = \begin{bmatrix} b_{33} & b_{35} & b_{36} \\ b_{53} & b_{55} & b_{56} \\ b_{63} & b_{65} & b_{66} \end{bmatrix} ;$$

and the b_{mn} are the linear coefficients in equation (1c).

From equation (4):

$$\{\sigma^{OP}\} = - [(B^*_{22})^{-1}(B^{*T}_{12})]\{\sigma^P\} \tag{4a}$$

and,

$$\begin{aligned} \{\varepsilon^P\} &= [B^*_{11}]\{\sigma^P\} + [B^*_{12}]\{\sigma^{OP}\} \\ &= [B^*_{11} - (B^*_{12})(B^*_{22})^{-1}(B^{*T}_{12})]\{\sigma^P\} . \end{aligned} \tag{4b}$$

In the plane strain problem the strain components only are in-plane: the state of stress is three-dimensional with the out-of-plane components given by equation (4a).

Alternately, equation (4b) is written:

$$\{\varepsilon^P\} = [B']\{\sigma^P\} \tag{4c}$$

where $[B']$ is a symmetric, 3 × 3 matrix of constants:

$$[B'] = [B^*_{11} - (B^*_{12})(B^*_{22})^{-1}(B^{*T}_{12})] ,$$

the determination of which characterizes a linear, anisotropic material in plane strain.

For perfectly isotropic materials submitted to a state of plane strain:

$$B'_{11} = \frac{1-\nu^2}{E} = B'_{22}$$

$$B'_{12} = -\frac{\nu}{1-\nu} = B'_{11}$$

$$B'_{13} = B'_{23} = 0$$

$$B'_{33} = \frac{2}{1-\nu} B'_{11}$$

where E and ν are Young's modulus and Poisson's ratio, respectively, and B'_{11}, B'_{12}, B'_{13}, B'_{22}, B'_{23}, and B'_{33} are the six elastic constants forming $[B']$.

4. Characterization of Anisotropic Materials in Planar Elasticity

For all planar problems in elasticity it is generally observed that the two-dimensional behavior of the material is defined by a constitutive law of in-plane components: a vector of in-plane components is related to a second vector of in-plane components by a symmetric, 3×3 matrix of elastic constants. The two cases of planar elasticity defined by equation (2c) and equation (4c) are not mathematically equivalent: the matrix of material constants $[A']$ is not the inverse of the matrix of material constants $[B']$. Equations (2c) and (4c) do permit, however, the writing of a general constitutive law applicable to all planar elasticity problems:

$$\{\sigma^P\} = [D]\{\varepsilon^P\} \tag{5}$$

where $[D]$ is a 3×3, symmetric matrix equal to matrix $[A']$ for plane stress

problems and equal to $[B']^{-1}$ for plane strain problems. It is the form of equation (5) with which this study is concerned: a technique of measuring the six, independent elastic constants,

$$[D] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ (\text{symmetric}) & & C_{33} \end{bmatrix},$$

is the goal of this study in the characterization of linear, anisotropic material constants.

B. FINITE ELEMENT FORMULATION

Analysis of the basic unit of a structure is the first step in the total analysis of the structure. The basic part under consideration here is the finite element. Attention is focused on problems of planar stress, although the technique is expressed in a form generally applicable to any linear, anisotropic material. The finite elements employed are necessarily restricted to two-dimensional configurations.

1. Finite Element Geometry

A myriad of two-dimensional shapes have been used in finite element analyses [5, 21], the simplest being that of a triangle. Another very simple shape is that of a rectangle. Other shapes, from trapezoids through to curved shapes, are also available. For the purpose of the characterization of a linear, anisotropic material the simple rectangle has been selected primarily because of its inherent symmetries and ease of mathematical description.

2. Displacement Formulation for the Finite Element

The displacement formulation for a finite element is well documented [4, 5, 17, 18, 21]. Appendix A presents the details for the formulation of a 4-noded, linear element.

The elemental stiffness is given by:

$$[k]^e = \iint_s [B]^T[D][B]ds \ , \tag{6}$$

where $[k]^e$ is defined as the elemental stiffness matrix, and the indicated integration is performed across the area of the element.

The equation of static equilibrium for an element can be written as:

$$\{Q\}^e = [k]^e\{\delta\}^e \ , \tag{7}$$

where $\{Q\}^e$ is the vector of forces acting at the elemental nodes.

Appendix A provides the specific derivation of the elemental stiffness matrix $[k]^e$ for a 4-noded rectangular finite element in closed-form. The resulting symmetric, 8 × 8 matrix is shown as Figure 7 of that appendix.

In conventional finite element analysis, elemental stiffness contributions are summed at common nodal points throughout the structure by the method of direct stiffness. Forces are summed throughout the structure, and it is observed that only forces external to the structure remain because of the nullifying effects of internal nodal forces. Resultant forces are of two kinds: forces due to the reactions of imposed physical boundary conditions and forces loading the structure. The equation of equilibrium for the structure is:

$$\{R\} = [K]\{\delta\} \ , \tag{8}$$

where $\{R\}$ is the vector of resultant forces acting at each node, generally equal to zero at internal nodes of the structure; $[K]$ is the master stiffness matrix; and $\{\delta\}$ is the vector of displacements containing the displacements at each node of the structure.

Equation (6) defines elemental stiffness as a function of the particular geometry of the element and the six elastic constants. Performing the matrix multiplication indicated by equation (7) and factoring-out the vector of elastic constants, a modified elemental stiffness matrix is defined [12, 13]:

$$\{Q\}^e = [k^*]^e\{C\} , \tag{9}$$

where $[k^*]$ is the modified elemental stiffness matrix and $\{C\}$ is the vector of elastic constants, C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, and C_{33}. This modified elemental stiffness matrix is not square: for the 4-noded element the modified stiffness matrix is of size 8×6.

The members of the modified elemental stiffness matrix are functions of the particular geometry of the element and the nodal displacements, u_i and v_i, of the element. Figure 8, Appendix A, presents the closed-form solution for the modified stiffness matrix for the 4-noded rectangular element. Such a closed-form solution for the modified stiffness matrix is notationally straight forward, but a good deal of tedious manipulation is hidden behind the symbols: closed-form solutions have not been successfully obtained for the elemental stiffness matrix or the modified elemental stiffness matrix for an 8-noded rectangular element because of repeated algebraic difficulties.

A simplified direct summation procedure is used to form the master modified stiffness matrix, and the forces are again summed across the structure:

$$\underset{n \times 1}{\{R\}} = \underset{n \times 6}{[K^*]} \; \underset{6 \times 1}{\{C\}} \quad . \tag{10}$$

3. <u>Material Characterization Using Finite Elements</u>

Figure 8, Appendix A, presents the 8×6 modified stiffness matrix derived for the 4-noded rectangular element. A structure composed of a number of these elements will exhibit a master modified stiffness matrix of order $n \times 6$. For the planar problems of elasticity two degrees of freedom exist at each node in the structure, and n equals twice the number of joints.

The over-determined system of equations (10) can be solved as follows:

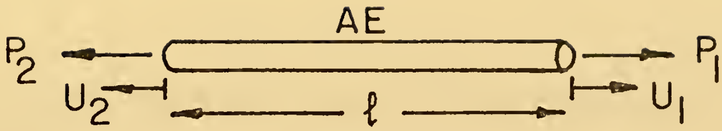
$$[K^*]^T\{R\} = [K^*]^T[K^*]\{C\}$$

$$([K^*]^T[K^*])^{-1}[K^*]^T\{R\} = \{C\} \quad , \tag{11}$$

if the product $[K^*]^T[K^*]$ is non-singular.

C. CHARACTERIZATION USING FINITE ELEMENTS: AN EXAMPLE

Figure 1 presents an illustrative example of the use of finite elements in the characterization of elastic solids and is a modification after Kavanagh [12]. The problem is restricted to one-dimension and to one elastic constant, E.



TYPICAL FINITE ELEMENT, ONE-DIMENSIONAL.

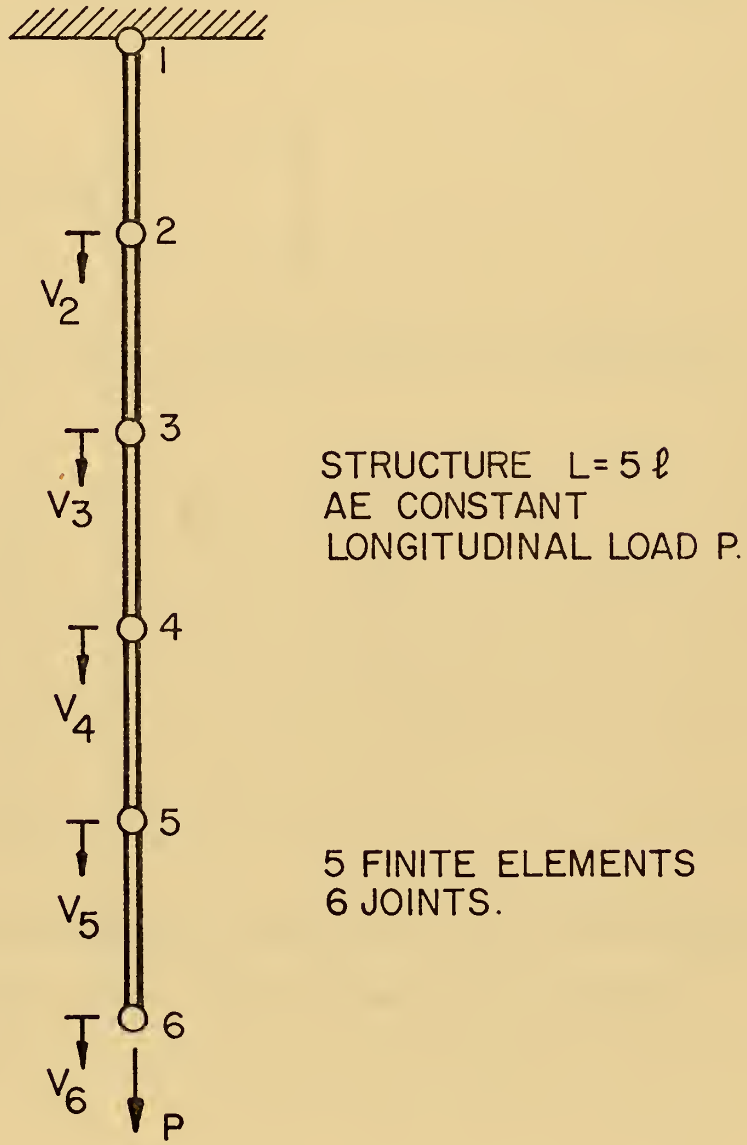


FIGURE 1 : ONE-DIMENSIONAL EXAMPLE PROBLEM ILLUSTRATING THE FINITE ELEMENT CHARACTERIZATION OF ELASTIC SOLIDS.

The example consists of a weight P, hanging from a slender rod of total length L and cross-sectional area A. The rod is divided into five finite elements, each element containing two nodes. There are a total of six structural joints, the first of which is constrained by a physically pinned connection.

The finite element in one-dimension yields an elemental stiffness matrix:

$$[k]^e = \frac{AE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} .$$

The master stiffness matrix for the overall structure is found by the direct stiffness method:

$$[K] = \frac{AE}{\ell} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} .$$

Imposition of the boundary condition at the first joint allows eliminating the first row and the first column, and the equation for structural equilibrium is written:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \end{Bmatrix} = \frac{AE}{\ell} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix} .$$

Performing the indicated matrix multiplication and factoring the material constant, E:

$$\underset{(5\times 1)}{\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P \end{Bmatrix}} = \underset{(5\times 1)}{\begin{bmatrix} (2v_2 - v_3)\ A/\ell \\ (-v_2 + 2v_3 - v_4)\ A/\ell \\ (-v_3 + 2v_4 - v_5)\ A/\ell \\ (-v_4 + 2v_5 - v_6)\ A/\ell \\ (-v_5 + v_6)\ A/\ell \end{bmatrix}} \underset{(1\times 1)}{\{E\}} .$$

Setting $A = 1.0$, $\ell = 100.0$, and $P = 10.0$, assume that the measured deflections are $v_6 = 5.0$, $v_5 = 4.0$, $v_4 = 3.0$, $v_3 = 2.0$, $v_2 = 1.0$:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10.0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.01 \end{Bmatrix} \{E\} .$$

Solving the equation for $\{E\}$:

$$[0 \quad 0 \quad 0 \quad 0 \quad 0.01] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10.0 \end{Bmatrix} = [0 \quad 0 \quad 0 \quad 0 \quad 0.01] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.01 \end{Bmatrix} \{E\}$$

$$0.10 = 0.0001 \{E\}$$

$$1000.0 = \{E\} \quad .$$

It is instructive to note that displacements calculated using this value of E would be identically those values assumed at joints 2, 3, 4, 5, and 6. In an actual experiment, however, the displacements obtained at the joints by measurement of the structure would be in error by some experimental amount. If E were known exactly, the E calculated would not be exact.

III. EXPERIMENTAL SIMULATION AND PROCEDURE

A. NODAL DISPLACEMENTS

The application of the finite element method to the characterization of elastic solids is predicated on the accurate knowledge of displacements at the nodes of the structure. Small errors in obtaining displacements will cause large errors to be introduced in the values of the calculated material constants, as reported by Kavanagh [12]. It has been necessary, therefore, to employ a simulation scheme to generate the displacements associated with a given loading configuration.

Appendix B consists of a listing of the digital computer program DPLISOP. DPLISOP is a double-precision version of PLISOP [3], a general purpose finite element program for planar problems in elasticity. DPLISOP was used to generate the nodal displacements required throughout this study.

1. Convergence Study

A convergence study using DPLISOP was performed. A steel plate, 10.0 inches square and 0.10 inches thick, was simulated in a state of plane stress. The top edge of the plate was constrained, and the bottom edge of the plate was uniformly loaded to a static load of 15.0 kips. Loading at the bottom edge nodes was simulated by the usual application of the consistent load vector concept [21]. Young's modulus and Poisson's ratio were taken to be $E = 29.50 \times 10^3$ ksi and $\nu = 0.287$, respectively. Finite element discretizations of one, four, and sixteen elements were used. Figure 2 shows these trial discretizations using the 4-noded element. Although DPLISOP is capable of utilizing 4-noded, 8-noded, and

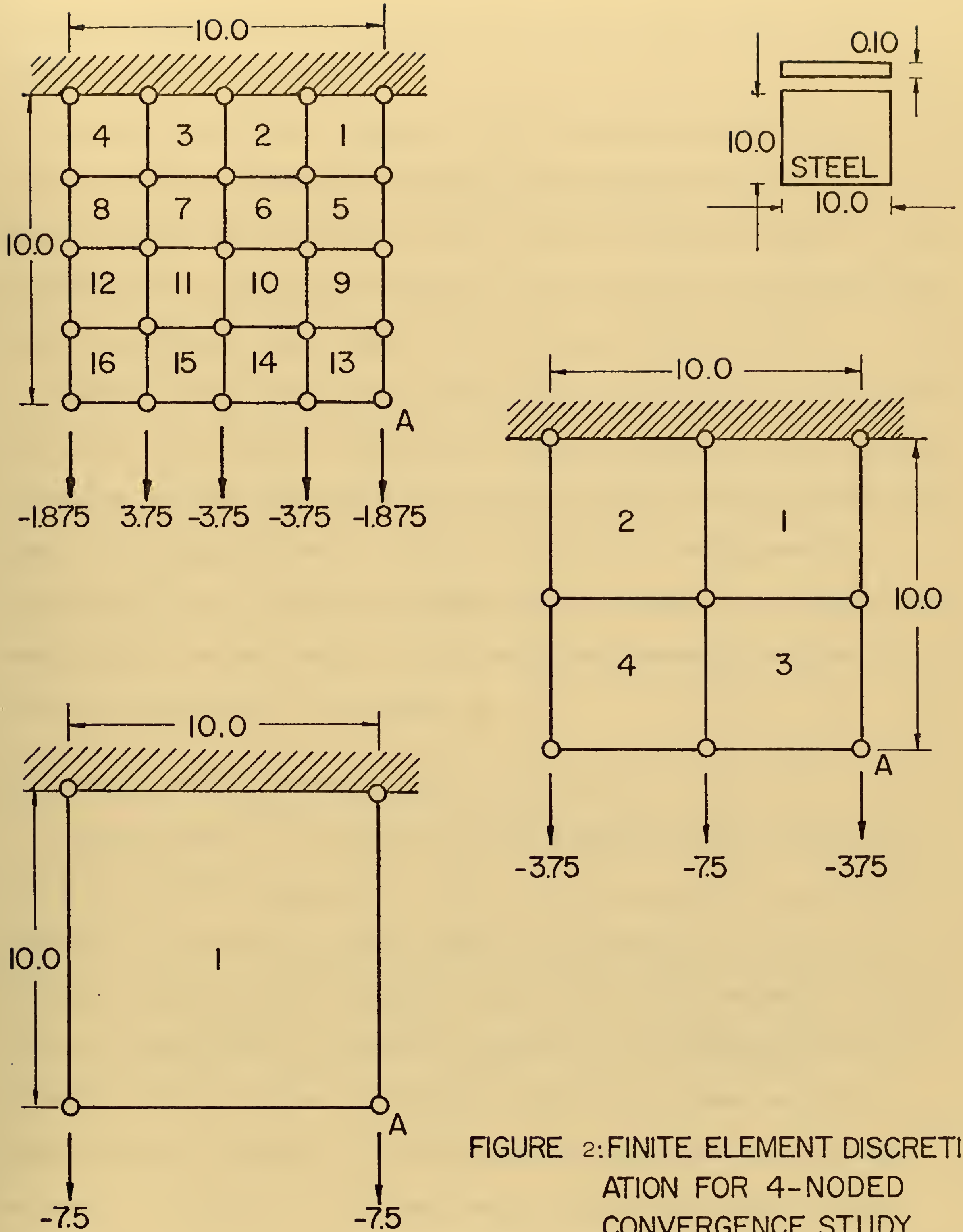


FIGURE 2: FINITE ELEMENT DISCRETIZATION FOR 4-NODED CONVERGENCE STUDY. STEEL PLATE, 10 BY 10, 0.10 THICK, TOTAL 15 KIP LOAD UNIFORMLY ALONG BOTTOM OF PLATE.

12-noded rectangular finite elements, runs were made for only the 4-noded and 8-noded elements.

Figure 3 presents the results of the convergence study for the 4-noded rectangular element, at Node A of the simulation structure. Similar results were obtained for the 8-noded rectangular element. The minimum number of elements necessary to ensure accurate displacements in both the horizontal and vertical directions was found to be four.

Implicit in the convergence study was the requirement that the experimental structure exhibit symmetries in both the horizontal and vertical directions and that elements be very nearly square to avoid geometrically induced anisotropic behavior. Figure 4 details the configuration, consisting of four, 4-noded rectangular finite elements, containing nine nodal points, and an overall 10.0 inches square geometry, selected for use in the remainder of the study.

B. COMPUTER PROGRAM DLASTIC

Program DLASTIC was written as a test program for the determination of the six elastic constants C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, and C_{33}, by direct inversion of the finite element formulation (equations (10) and (11)), in order to characterize a general, linear anisotropic material. Double-precision displacements generated in DPLISOP were used as input data to DLASTIC and the six material constants calculated. These calculated numerical values proved to be exactly the characterizing six material constants of equation (3a). These results appear as a representative output which, with the listing of DLASTIC, form Appendix C.

Element geometry used in DLASTIC is restricted to the 4-noded rectangular finite element shape.

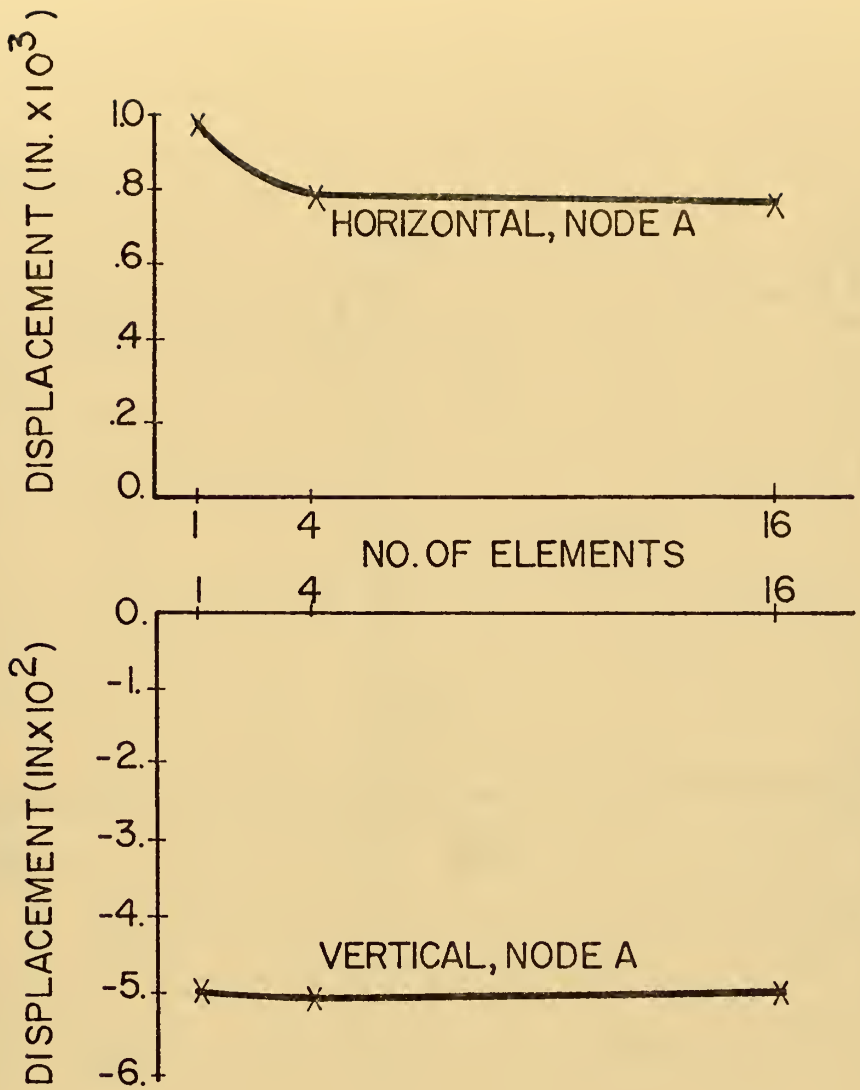


FIGURE 3: CONVERGENCE STUDY, NODE-A, USING 4-NODED RECTANGULAR FINITE ELEMENTS. STEEL PLATE IS 10 BY 10 SQUARE, 0.10 THICK, LOADED IN PLANE STRESS BY 15 KIP TOTAL LOAD UNIFORMLY DISTRIBUTED ALONG BOTTOM EDGE OF PLATE.

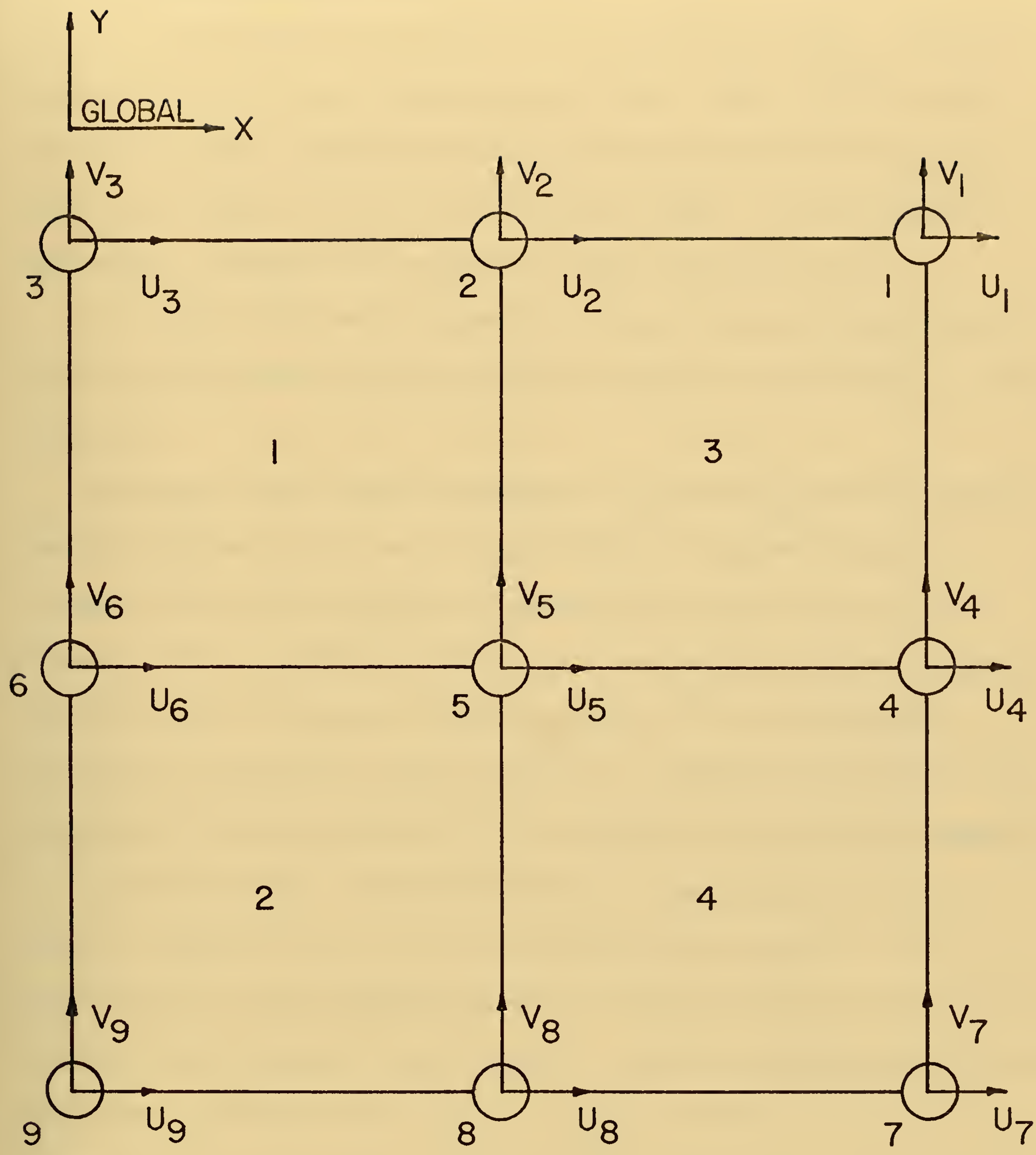


FIGURE 4 : 4-NODED 4-ELEMENT STRUCTURE SHOWING JOINT NUMBERING CONVENTION AND POSITIVE DISPLACEMENTS AT JOINTS.

1. Program Organization

Program DLASTIC was written as a modular algorithm: the MAIN program calls, in order, the subroutines INPUT, MERGE, BMULT, SINGL, INVRT, and ANSER. Subroutine MERGE, in turn, calls subroutine STIFF. Subroutine SINGL, in turn, calls subroutine EIGEN. Communication is maintained throughout the program by common blocks of storage.

The eight digit numbers appearing in the discussion which follows refer to the program sequencing numbers appearing in the program listing, Appendix C.

Subroutine INPUT (00000700-00001350) is called by the MAIN program. Punched data cards are read and echo-checked. Required input data includes structural parameter specifications, such as the number of elements and joints (NEL and NJT), nodal connectivity for every element (NCON), and the coordinates (COORD), forces (R), and displacements (U and V) for each joint in the structure. Specific directions for data preparation have been included within the MAIN program listing (Appendix C).

Subroutine MERGE (00001360-00001820) is next called by the MAIN program. MERGE forms the master modified stiffness matrix $[K^*]$, by superposition of the elemental modified stiffness matrices at appropriate common joints throughout the structure. Subroutine STIFF (00001830-00002910) is called in MERGE within a do-loop sequence indexed over the number of elements in the structure. Subroutine STIFF forms the elemental modified stiffness matrix.

Next called by the MAIN program is subroutine BMULT (00002920-00003270). In BMULT, the master modified stiffness matrix, BK, is premultiplied by

its transpose, BKT. The resulting symmetric matrix has been designated BSYM. The vector of nodal forces, R, is also premultiplied by BKT: this result has been designated BR.

The first multiplication in subroutine BMULT, the symmetric matrix BSYM, corresponds to the term $[K^*]^T[K^*]$ of equation (11). The MAIN program calls subroutine SINGL (00003280-00003530), which calls subroutine EIGEN (00003540-00005350) and a determination is made regarding the singularity or non-singularity of the matrix BSYM.

EIGEN calculates the six eigenvalues of BSYM associated with the as yet undetermined six material constants. SINGL causes the six eigenvalues to be printed as part of the program output, and inspection of this portion of the output is required by the program user to ascertain the validity of the total program output. Discussion of this requirement is deferred to the section following this discussion of program organization.

Subroutine INVRT (00005360-00006930) is next called from the MAIN program. The result of the first multiplication in subroutine BMULT, the matrix BSYM, is inverted using the Gauss-Jordan method. The determinant calculated in INVRT is ignored, for reasons which will become clear in the discussion of the eigenvalue criterion.

The vector of material constants, C, is calculated in subroutine ANSER (00006940-00007190) by premultiplying the vector BR, calculated in subroutine BMULT, by the inverted matrix BSYM, which occupies the storage location previously occupied by the non-inverted matrix. The resulting vector C, if valid, contains the six material constants C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, and C_{33}. ANSER causes the constants to be printed as program output.

DLASTIC was written in FORTRAN IV and compiled with the G-compiler under release 18 on an IBM 360/67 machine. It required 84,000 bytes of core and used approximately one-second CPU time for a typical run.

2. Eigenvalue Criterion of Matrix Singularity

A matrix is said to be singular when the determinant of the matrix is equal to zero. However, when a matrix is ill-conditioned and nearly singular the numerical evaluation of the determinant is almost invariably completely unreliable. Round-off in most algorithms is sufficient to introduce an artificial round-off value of zero that invalidates the calculation of the determinant.

The calculation of the eigenvalues of a symmetric matrix is always a well conditioned numerical problem, even when the matrix is exactly singular. A better indication of singularity and near-singularity is obtained by examining all eigenvalues appropriate to the matrix. The condition number of the matrix is then obtained as the ratio of the largest absolute value of the eigenvalues to the smallest absolute value of the eigenvalues. If this condition number is greater than 10^{12} the matrix is too nearly singular to be inverted accurately.

In program DLASTIC it was necessary to employ the eigenvalue criterion of singularity. The symmetric matrix BSYM, formed in subroutine BMULT, was found to exhibit eigenvalues in the range 10^{-6} to 10^{-9}. The condition number is formed, 10^{3}, and it is obvious that matrix BSYM is very far from any singularity.

C. DISPLACEMENT PERTURBATIONS

An exact characterization of an elastic solid was presented in a previous section (see the representative output, Appendix C). Such a

result utilized the full double-precision displacement data (sixteen digits) generated within the simulation program DPLISOP. Such exhaustive displacement data cannot be expected to be available from a physical experiment or possible test device. It was necessary, therefore, to perform a series of perturbation analyses in order to ascertain the lower limit of displacement data accuracy required for the characterization of elastic solids by the direct inversion of the finite element formulation.

A first set of truncated displacements was used to determine how sensitive the elastic constants would be to such approximations. Round-off to five, six, seven, and eight places of decimals was used (see Table 1). The elastic constants were calculated for each set of rounded displacements using program DLASTIC. The relative errors of the resulting material constants were then evaluated.

The results for the material constants and the associated errors are presented as Table 2.

Random perturbations, ranging in value from -5 to +5, were next applied to the rounded results in the fifth, sixth, seventh, and eighth places of decimals. Five different sets of displacements were obtained for each of the rounded results shown in Table 1.

Every set of displacements was used as input data for program DLASTIC, and all six material constants were obtained for each case. The results are shown in Tables 3 and 4.

JOINT		DISPLACEMENT	NON-RANDOM DISPLACEMENT PERTURBATIONS			
4	u	-0.748329139132374D-03	-0.00075	-0.000748	-0.0007483	-0.00074833
	v	-0.252610022185014D-02	-0.00253	-0.002526	-0.0025261	-0.00252610
5	v	-0.235459058006389D-02	-0.00253	-0.002355	-0.0023546	-0.00235459
6	u	0.748329139132375D-03	0.00075	0.000748	0.0007483	0.00074833
	v	-0.252610022185015D-02	-0.00253	-0.002526	-0.0025261	-0.00252610
7	u	-0.770070823585924D-03	-0.00077	-0.000770	-0.0007701	-0.00077007
	v	-0.500613331025850D-02	-0.00501	-0.005006	-0.0050061	-0.00500613
8	v	-0.497625861993874D-02	-0.00498	-0.004976	-0.0049763	-0.00497626
9	u	0.770070823585927D-03	0.00077	0.000770	0.0007701	0.00077007
	v	-0.500613331025850D-02	-0.00501	-0.005006	-0.0050061	-0.00500613
			RUN 1	RUN 2	RUN 3	RUN 4

TABLE 1 : NON-RANDOM PERTURBATIONS (SIMPLE ROUNDING) OF NON-ZERO HORIZONTAL AND VERTICAL NODAL DISPLACEMENTS, SIMULATED 10.0 BY 10.0, 0.10 THICK STEEL STRUCTURE IN PLANE STRESS. 15.0 KIPS LOAD.

EXACT VALUES, ANISOTROPIC (PL. STRESS)					
C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}
32147.998	9226.476	0.0	32147.998	0.0	11460.0

RUN 1

		ERROR, %
C_{11}	32757.	1.9
C_{12}	9381.	1.7
C_{13}	0.	0.0
C_{22}	32159.	0.0
C_{23}	0.	0.0
C_{33}	11421.	0.3

RUN 2

		ERROR, %
C_{11}	32093.	0.2
C_{12}	9211.	0.2
C_{13}	0.	0.0
C_{22}	32145.	0.0
C_{23}	0.	0.0
C_{33}	11471.	0.0

RUN 3

		ERROR, %
C_{11}	32145.	0.0
C_{12}	9226.	0.0
C_{13}	0.	0.0
C_{22}	32148.	0.0
C_{23}	0.	0.0
C_{33}	11463.	0.0

RUN 4

		ERROR, %
C_{11}	32148.	0.0
C_{12}	9226.	0.0
C_{13}	0.	0.0
C_{22}	32148.	0.0
C_{23}	0.	0.0
C_{33}	11461.	0.0

TABLE 2: THE SIX ANISOTROPIC CONSTANTS CALCULATED IN PROGRAM DLASTIC, BY ROUNDING OF DISPLACEMENT DATA.

	EXACT	1	2	3	4	5	PERTURBATION 5 Percent error				
C_{11}	32147.998	39194.	35217.	31973.	29993.	36222.	22.	9.5	0.5	6.7	13.
C_{12}	9226.476	10978.	9954.	9243.	8993.	10353.	19.	7.9	0.2	2.5	12.
C_{13}	0.0	1041.	620.	328.	-759.	-414.	--	--	--	--	--
C_{22}	32147.998	32332.	32402.	32352.	32261.	32401.	0.6	0.8	0.6	0.4	0.8
C_{23}	0.0	95.	52.	-5.	-53.	-61.	--	--	--	--	--
C_{33}	11460.761	10665.	9882.	11924.	11314.	8591.	6.9	14.	4.0	1.3	25.

							PERTURBATION 6 Percent error				
C_{11}	32147.998	31290.	31660.	31553.	32201.	32136.	2.7	1.5	1.9	0.2	0.0
C_{12}	9226.476	9005.	9108.	9058.	9237.	9214.	2.4	1.3	1.8	0.1	0.1
C_{13}	0.0	32.	-105.	75.	-13.	-13.	--	--	--	--	--
C_{22}	32147.998	32108.	32115.	32130.	32174.	32136.	0.1	0.1	0.0	0.0	0.0
C_{23}	0.0	13.	-18.	6.	-7.	4.	--	--	--	--	--
C_{33}	11460.761	11594.	11518.	11699.	11481.	11618.	1.2	0.5	2.1	0.2	1.4

TABLE 3: THE SIX ANISOTROPIC MATERIAL CONSTANTS CALCULATED IN PROGRAM DLASTIC, BASED ON RANDOM PERTURBATIONS TO THE FIFTH AND SIXTH DECIMAL PLACES OF DISPLACEMENTS.

	EXACT	1	2	3	4	5	PERTURBATION 7 Percent error				
C_{11}	32147.998	32153.	32102.	32074.	32094.	32103.	0.0	0.1	0.2	0.2	0.1
C_{12}	9226.476	9227.	9215.	9211.	9215.	9214.	0.0	0.1	0.2	0.1	0.1
C_{13}	0.0	-6.	-3.	-9.	-14.	-4.	--	--	--	--	--
C_{22}	32147.998	32146.	32144.	32145.	32145.	32143.	0.0	0.0	0.0	0.0	0.0
C_{23}	0.0	-2.	-2.	-1.	-2.	-1.	--	--	--	--	--
C_{33}	11460.761	11447.	11467.	11465.	11458.	11468.	0.1	0.0	0.0	0.0	0.0
							PERTURBATION 8 Percent error				
C_{11}	32147.998	32142.	32154.	32145.	32155.	32141.	0.0	0.0	0.0	0.0	0.0
C_{12}	9226.476	9225.	9228.	9226.	9228.	9225.	0.0	0.0	0.0	0.0	0.0
C_{13}	0.0	0.	0.	0.	-1.	-1.	0.0	0.0	0.0	--	--
C_{22}	32147.998	32148.	32148.	32148.	32148.	32147.	0.0	0.0	0.0	0.0	0.0
C_{23}	0.0	0.	0.	0.	0.	0.	0.0	0.0	0.0	0.0	0.0
C_{33}	11460.761	11463.	11459.	11459.	11462.	11463.	0.0	0.0	0.0	0.0	0.0

TABLE 4: THE SIX ANISOTROPIC MATERIAL CONSTANTS CALCULATED IN PROGRAM DLASTIC, BASED ON RANDOM PERTURBATIONS TO THE SEVENTH AND EIGHTH DECIMAL PLACES OF DISPLACEMENTS.

IV. RESULTS AND CONCLUSIONS

The six, anisotropic material constants appropriate to the plane stress elasticity problem and the particular material were calculated in equation (3a). Using the full double-precision displacement data generated by simulating an elastic structure in a state of plane stress program DLASTIC was used to calculate numerical values for these same six, anisotropic constants. Both sets of numerical constants were identical. For the purpose of discussion these values for the six material constants have been considered the exact numerical values of the constants.

Program DLASTIC required closed-form solutions to the stiffness matrices of the finite element formulation. Closed-form solutions for the elemental stiffness matrix, $[k]$, and the modified elemental stiffness matrix, $[k^*]$, were successfully obtained for the 4-noded, rectangular finite element. These results form Figures 7 and 8, Appendix A.

A. PERTURBATION RESULTS

Reasonable numerical values for the six elastic constants were obtained, based on approximate displacement data generated by rounding to as few as five decimal places. Table 2 presents the calculated values for these constants. Eigenvalues were also calculated: the four runs using simple roundings as approximations returned eigenvalues in the range 10^{-6} to 10^{-9}. Approximate displacement data to the fifth decimal place caused the returned numerical values of the constants to be within two percent of the exact numerical values, and approximate

displacement data to the sixth, seventh, and eighth decimal places resulted in numerical values within half-of-one percent of the exact values of the material constants.

Twenty sets of displacement data were generated for the random analyses. All twenty data sets were run in program DLASTIC and numerical values for the elastic constants calculated.

The numerical values for the calculated elastic constants are presented in Tables 3 and 4. The calculated eigenvalues for all random perturbations were within the range of 10^{-6} to 10^{-9}. Comparison of these calculated material constants to the exact values gives an idea of the sensitivity of the elastic constants to the accuracy of the displacement data.

Random perturbations applied to the results rounded to five decimal places gave the largest errors in the elastic constants. Worse values were consistently the calculated constant C_{13}, the values of which ranged from -759.5 ksi to 1041.0 ksi.

Simulated structural displacements accurate to 10^{-5} inches, the sixth decimal place of displacements being random, returned numerical values for the material constants generally less than three percent in error. Values of C_{13}, though not identically zero, were sufficiently small to consider them calculated zero except in two cases. Larger errors were experienced in the calculated values for C_{11} and C_{12} for all five sets.

Actual displacement measurements for a test device or physical experiment could be expected to require a horizontal displacement data precision of one part in eight hundred (1/800) and a vertical

displacement precision of one part in five hundred (1/500). These precisions could be expected to yield numerical values for the material constants in the range of two to three percent error.

Perturbations to the seventh and eighth decimal places returned material constants less than one percent in error (Table 4). An actual test device would require a horizontal measurement precision of one part in eight thousand (1/8000) and a vertical precision of one part in five thousand (1/5000), based on the simulated structural displacements being accurate to 10^{-6} inches and the seventh decimal place of displacements being random. The test device could be expected to yield numerical values for the material constants much less than one percent in error.

The numerical values returned from perturbations of displacement data to the eighth decimal place indicated required precisions of one part in eighty thousand (1/80000), horizontal, and one part in fifty thousand (1/50000), vertical. These precisions could be expected to yield numerical values for the material constants much less than one percent in error.

B. CONCLUSIONS

As a consequence of this study of the finite element technique, as utilized in the characterization of linear, anisotropic elastic solids, several conclusions can be drawn:

1. The finite element formulation under study can be applied to the mechanical characterization of linear, anisotropic solids.

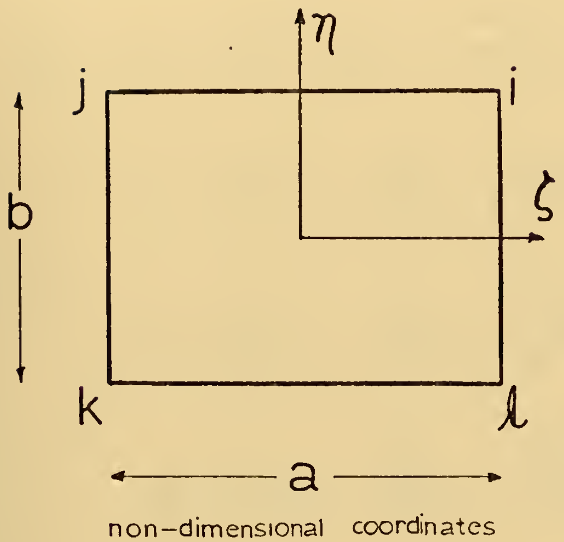
2. The accuracy of characterization is dependent on the sensitivity of the finite element formulation.

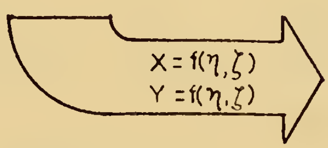
3. The sensitivity of the finite element formulation is dependent on the precision with which displacement data is measured.

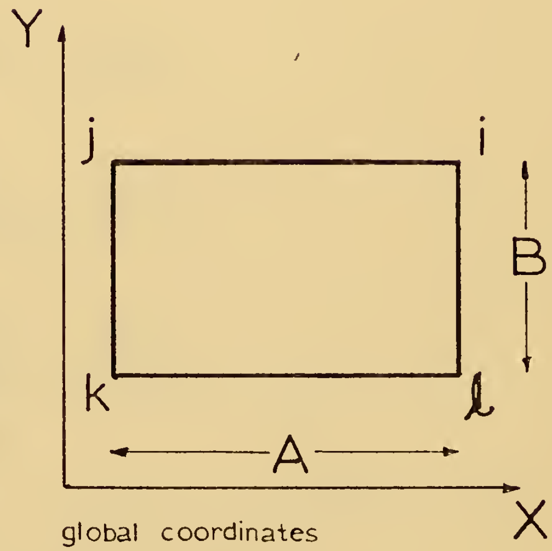
4. A device designed to determine the six linear, anisotropic elastic constants to an accuracy of three percent will require a precision in displacement measurements of at least one part in eight hundred (1/800).

5. A device designed to determine the six linear, anisotropic elastic constants to an accuracy of one percent will require a precision in displacement measurements of at least one part in eight thousand (1/8000).

Derivation of stiffness matrices for a 4-noded finite element







$$N_r = \frac{1}{4}(1+\zeta_o)(1+\eta_o)$$

$$\zeta_o = \zeta_r \zeta \qquad \zeta_r = \pm 1$$

$$\eta_o = \eta_r \eta \qquad \eta_r = \pm 1$$

$$N_i = \frac{1}{4}(1+\zeta)(1+\eta)$$
$$N_j = \frac{1}{4}(1-\zeta)(1+\eta)$$
$$N_k = \frac{1}{4}(1-\zeta)(1-\eta)$$
$$N_\ell = \frac{1}{4}(1+\zeta)(1-\eta)$$

$$x = N_i x_i + N_j x_j + N_k x_k + N_\ell x_\ell$$
$$y = N_i y_i + N_j y_j + N_k y_k + N_\ell y_\ell$$

$$\frac{\partial N_i}{\partial \zeta} = \frac{1}{4}(1+\eta) \quad \frac{\partial N_i}{\partial \eta} = \frac{1}{4}(1+\zeta) \quad \frac{\partial N_j}{\partial \zeta} = -\frac{1}{4}(1+\eta) \quad \frac{\partial N_j}{\partial \eta} = \frac{1}{4}(1-\zeta)$$

$$\frac{\partial N_k}{\partial \zeta} = \frac{1}{4}(\eta-1) \quad \frac{\partial N_k}{\partial \eta} = \frac{1}{4}(\zeta-1) \quad \frac{\partial N_\ell}{\partial \zeta} = \frac{1}{4}(1-\eta) \quad \frac{\partial N_\ell}{\partial \eta} = -\frac{1}{4}(1+\zeta)$$

$$\frac{\partial x}{\partial \zeta} = \frac{\partial N_i}{\partial \zeta} x_i + \frac{\partial N_j}{\partial \zeta} x_j + \frac{\partial N_k}{\partial \zeta} x_k + \frac{\partial N_\ell}{\partial \zeta} x_\ell$$

$$\frac{\partial x}{\partial \eta} = \frac{\partial N_i}{\partial \eta} x_i + \frac{\partial N_j}{\partial \eta} x_j + \frac{\partial N_k}{\partial \eta} x_k + \frac{\partial N_\ell}{\partial \eta} x_\ell$$

$$\frac{\partial y}{\partial \zeta} = \frac{\partial N_i}{\partial \zeta} y_i + \frac{\partial N_j}{\partial \zeta} y_j + \frac{\partial N_k}{\partial \zeta} y_k + \frac{\partial N_\ell}{\partial \zeta} y_\ell$$

$$\frac{\partial y}{\partial \eta} = \frac{\partial N_i}{\partial \eta} y_i + \frac{\partial N_j}{\partial \eta} y_j + \frac{\partial N_k}{\partial \eta} y_k + \frac{\partial N_\ell}{\partial \eta} y_\ell$$

Define the Jacobian, [J]:

$$[J(\eta,\zeta)] = \begin{bmatrix} \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial \zeta} & \frac{\partial N_j}{\partial \zeta} & \frac{\partial N_k}{\partial \zeta} & \frac{\partial N_\ell}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} & \frac{\partial N_j}{\partial \eta} & \frac{\partial N_k}{\partial \eta} & \frac{\partial N_\ell}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_i & y_i \\ x_k & y_j \\ x_k & y_k \\ x_\ell & y_\ell \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{a}$$

Define the displacement field, $\{\delta\} = \begin{Bmatrix} u \\ v \end{Bmatrix}$

$$u = N_i u_i + N_j u_j + N_k u_k + N_\ell u_\ell$$

$$v = N_i v_i + N_j v_j + N_k v_k + N_\ell v_\ell \qquad \text{(b)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial N_i}{\partial x} u_i + \frac{\partial N_j}{\partial x} u_j + \frac{\partial N_k}{\partial x} u_k + \frac{\partial N_\ell}{\partial x} u_\ell$$

$$\frac{\partial u}{\partial y} = \frac{\partial N_i}{\partial y} u_i + \frac{\partial N_j}{\partial y} u_j + \frac{\partial N_k}{\partial y} u_k + \frac{\partial N_\ell}{\partial y} u_\ell$$

$$\frac{\partial v}{\partial x} = \frac{\partial N_i}{\partial x} v_i + \frac{\partial N_j}{\partial x} v_j + \frac{\partial N_k}{\partial x} v_k + \frac{\partial N_\ell}{\partial x} v_\ell$$

$$\frac{\partial v}{\partial y} = \frac{\partial N_i}{\partial y} v_i + \frac{\partial N_j}{\partial y} v_j + \frac{\partial N_k}{\partial y} v_k + \frac{\partial N_\ell}{\partial y} v_\ell$$

In general,

$$\frac{\partial N_r}{\partial x} = \frac{\partial N_r}{\partial \zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial N_r}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$\frac{\partial N_r}{\partial y} = \frac{\partial N_r}{\partial \zeta} \frac{\partial \zeta}{\partial y} + \frac{\partial N_r}{\partial \eta} \frac{\partial \eta}{\partial y}$$

or,

$$\frac{\partial N_r}{\partial \zeta} = \frac{\partial N_r}{\partial x} \frac{\partial x}{\partial \zeta} + \frac{\partial N_r}{\partial y} \frac{\partial y}{\partial \zeta}$$

$$\frac{\partial N_r}{\partial \eta} = \frac{\partial N_r}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_r}{\partial y} \frac{\partial y}{\partial \eta} \qquad \text{(c)}$$

Thus,

$$\begin{Bmatrix} \dfrac{\partial N_r}{\partial \zeta} \\ \\ \dfrac{\partial N_r}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} \\ \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_r}{\partial x} \\ \\ \dfrac{\partial N_r}{\partial y} \end{Bmatrix}$$

$$= [J(\eta,\zeta)] \begin{Bmatrix} \dfrac{\partial N_r}{\partial x} \\ \\ \dfrac{\partial N_r}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial N_r}{\partial x} \\ \\ \dfrac{\partial N_r}{\partial y} \end{Bmatrix} = [J(\eta,\zeta)]^{-1} \begin{Bmatrix} \dfrac{\partial N_r}{\partial \zeta} \\ \\ \dfrac{\partial N_r}{\partial \eta} \end{Bmatrix} \qquad \text{(d)}$$

where, $$[J(\eta,\zeta)]^{-1} = 2 \begin{bmatrix} 1/A & 0 \\ 0 & 1/B \end{bmatrix} .$$

For each r, $r = (i, j, k, \ell)$:

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \dfrac{1+\eta}{A} \\ \\ \dfrac{1+\zeta}{B} \end{Bmatrix} \qquad \begin{Bmatrix} \dfrac{\partial N_j}{\partial x} \\ \\ \dfrac{\partial N_j}{\partial y} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \dfrac{-(1+\eta)}{A} \\ \\ \dfrac{(1-\zeta)}{B} \end{Bmatrix} \qquad \text{(e)}$$

$$\begin{Bmatrix} \dfrac{\partial N_k}{\partial x} \\ \\ \dfrac{\partial N_k}{\partial y} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \dfrac{(\eta-1)}{A} \\ \\ \dfrac{(\zeta-1)}{B} \end{Bmatrix} \qquad \begin{Bmatrix} \dfrac{\partial N_\ell}{\partial x} \\ \\ \dfrac{\partial N_\ell}{\partial y} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \dfrac{1-\eta}{A} \\ \\ \dfrac{-(1+\zeta)}{B} \end{Bmatrix} \tag{e}$$

In general, for any two-dimensional element:

$$[B_r] = \begin{bmatrix} \dfrac{\partial N_r}{\partial x} & 0 \\ 0 & \dfrac{\partial N_r}{\partial y} \\ \dfrac{\partial N_r}{\partial y} & \dfrac{\partial N_r}{\partial x} \end{bmatrix} \tag{f}$$

and $[B] = [B_i \quad B_j \quad B_k \quad B_\ell]$ for the 4-noded rectangular element.

Specifically:

$$[B] = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & 0 & \dfrac{\partial N_j}{\partial x} & 0 & \dfrac{\partial N_k}{\partial x} & 0 & \dfrac{\partial N_\ell}{\partial x} & 0 \\ 0 & \dfrac{\partial N_i}{\partial y} & 0 & \dfrac{\partial N_j}{\partial y} & 0 & \dfrac{\partial N_k}{\partial y} & 0 & \dfrac{\partial N_\ell}{\partial y} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_k}{\partial y} & \dfrac{\partial N_k}{\partial x} & \dfrac{\partial N_\ell}{\partial y} & \dfrac{\partial N_\ell}{\partial x} \end{bmatrix} \tag{g}$$

Substituting the functions from equation (e) into equation (g), the matrix of first partial derivatives of shape functions for a 4-noded rectangular finite element, [B], is formed, (Figure 5).

The matrix of elastic constants, for planar elasticity is:

$$[D] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \multicolumn{2}{c}{\text{(symmetric)}} & C_{33} \end{bmatrix} \qquad \text{(h)}$$

where C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, and C_{33} are the six elastic constants.

The products

$$[D][B], \quad \text{and}$$

$$[B]^T[D][B]$$

are then formed (Figure 6), where

$$d^2 = \frac{(1+\eta)^2}{A^2} \qquad h^2 = \frac{(1-\zeta)^2}{B^2} \qquad gf = \frac{(1-\eta)(1+\zeta)}{AB}$$

$$g^2 = \frac{(1+\zeta)^2}{B^2} \qquad df = \frac{(1-\eta)^2}{A^2} \qquad gh = \frac{(1-\zeta^2)}{B^2} \qquad \text{(i)}$$

$$f^2 = \frac{(1-\eta)^2}{A^2} \qquad dg = \frac{(1+\eta)(1+\zeta)}{AB} \qquad fh = \frac{(1-\eta)(1-\zeta)}{AB}$$

$$dh = \frac{(1+\eta)(1-\zeta)}{AB}$$

$$[B] = \frac{1}{2}\begin{bmatrix} \frac{1}{A}(1+\eta) & 0 & -\frac{1}{A}(1-\eta) & 0 & \frac{1}{A}(\eta-1) & 0 & \frac{1}{A}(1-\eta) & 0 \\ 0 & \frac{1}{B}(1+\xi) & 0 & \frac{1}{B}(1-\xi) & 0 & \frac{1}{B}(\xi-1) & 0 & -\frac{1}{B}(1+\xi) \\ \frac{1}{B}(1+\xi) & \frac{1}{A}(1+\eta) & \frac{1}{B}(1-\xi) & -\frac{1}{A}(1+\eta) & \frac{1}{B}(\xi-1) & \frac{1}{A}(\eta-1) & -\frac{1}{B}(1+\xi) & \frac{1}{A}(1-\eta) \end{bmatrix}$$

LET, $d = \frac{1}{A}(1+\eta)$

$f = \frac{1}{A}(1-\eta)$

$g = \frac{1}{B}(1+\xi)$

$h = \frac{1}{B}(1-\xi)$

THEN,

$$[B] = \frac{1}{2}\begin{bmatrix} d & o & -d & o & -f & o & f & o \\ o & g & o & h & o & -h & o & -g \\ g & d & h & -d & -h & -f & -g & f \end{bmatrix}$$

FIGURE 5: [B], THE MATRIX OF FIRST-PARTIAL DERIVATIVES OF SHAPE FUNCTIONS FOR A 4-NODED RECTANGULAR FINITE ELEMENT.

$$[D]\,[B] = \frac{1}{2}\begin{bmatrix} (c_{11}d+c_{13}g) & (c_{12}g+c_{13}d) & (-c_{11}d+c_{13}h) & (c_{12}h-c_{13}d) & (-c_{11}f-c_{13}h) & (-c_{12}h-c_{13}f) & (c_{11}f-c_{13}g) & (-c_{12}g+c_{13}f) \\ (c_{12}d+c_{23}g) & (c_{22}g+c_{23}d) & (-c_{12}d+c_{23}h) & (c_{22}h-c_{23}d) & (-c_{12}f-c_{23}h) & (-c_{22}h-c_{23}f) & (c_{12}f-c_{23}g) & (-c_{22}g+c_{23}f) \\ (c_{13}d+c_{33}g) & (c_{23}g+c_{33}d) & (-c_{13}d+c_{33}h) & (c_{23}h-c_{33}d) & (-c_{13}f-c_{33}h) & (-c_{23}h-c_{33}f) & (c_{13}f-c_{33}g) & (-c_{23}g+c_{33}f) \end{bmatrix}$$

AND,

$$[B]^T\,[D]\,[B] = \frac{1}{4}\begin{bmatrix} (c_{11}d^2+c_{13}gd+c_{13}gd+c_{33}g^2) & (c_{12}gd+c_{13}d^2+c_{23}g+c_{33}gd) & (-c_{11}d^2+c_{13}hd-c_{13}gd+c_{33}hg) & (c_{12}hd-c_{13}d^2+c_{23}hg-c_{33}dg) & (-c_{11}fd-c_{13}hd-c_{13}gf-c_{33}hg) & (-c_{12}hd-c_{13}fd-c_{23}gh-c_{33}gf) & (c_{11}fd-c_{13}gd+c_{23}g+c_{33}fg) & (-c_{12}gd+c_{13}fd-c_{23}g+c_{33}fg) \\ & (c_{22}g^2+c_{23}dg+c_{23}gd+c_{33}d^2) & (-c_{12}gd+c_{23}hg-c_{13}d+c_{33}hd) & (c_{22}hg-c_{23}dg+c_{23}hd-c_{33}d^2) & (-c_{12}fg-c_{23}gh-c_{13}fd-c_{33}hd) & (-c_{22}hg-c_{23}fg-c_{23}hd-c_{33}fd) & (c_{12}fg-c_{23}g^2+c_{13}fd\ c_{33}gd) & (-c_{22}g^2+c_{23}gf-c_{23}gd+c_{33}fd) \\ & & (c_{11}d^2-c_{13}hd-c_{13}dh+c_{33}h^2) & (-c_{12}hd+c_{13}d^2+c_{23}h-c_{33}dh) & (c_{11}fd+c_{13}hd-c_{13}fh-c_{33}h^2) & (c_{12}hd+c_{13}fd-c_{23}h^2-c_{33}fh) & (-c_{11}fd+c_{13}gd+c_{13}fh-c_{33}gh) & (c_{12}gd-c_{13}fd-c_{23}gh+c_{33}fh) \\ & & & (c_{22}h^2-c_{23}dh-c_{23}dh+c_{33}d^2) & (-c_{12}fh-c_{23}h^2+c_{13}fd+c_{33}hd) & (-c_{22}h^2-c_{23}fh+c_{23}hd+c_{33}fd) & (c_{12}fh-c_{23}gh-c_{13}fd+c_{33}gd) & (-c_{22}gh+c_{23}fh+c_{23}gd-c_{33}fd) \\ & & & & (c_{11}f^2+c_{13}hf+c_{13}fh+c_{33}h^2) & (c_{12}hf+c_{13}f^2+c_{23}h^2+c_{33}fh) & (-c_{11}f^2+c_{13}gf-c_{13}fh+c_{33}gh) & (c_{12}gf-c_{13}f^2+c_{23}gh-c_{33}fh) \\ & & & & & (c_{22}h+c_{23}fh+c_{23}hf+c_{33}f^2) & (-c_{12}fh+c_{23}gh-c_{13}f^2+c_{33}gf) & (c_{22}gh-c_{23}fh+c_{23}gf-c_{33}f^2) \\ & \text{(SYMMETRIC)} & & & & & (c_{11}f^2-c_{13}gf-c_{13}fg+c_{33}g^2) & (-c_{12}gf+c_{13}f^2+c_{23}g^2-c_{33}fg) \\ & & & & & & & (c_{22}g^2-c_{23}fg-c_{23}gf+c_{33}f^2) \end{bmatrix}$$

FIGURE 6

The elemental stiffness is defined:

$$[k]^e = \int_y \int_x [B]^T[D][B]dxdy \ . \qquad (j)$$

The product $[B]^T[D][B]$, Figure 6, and equation (i) are functions in (η,ζ). With $dxdy = \det [J(\eta,\zeta)]$:

$$[k]^e = \int_{-1}^{1} \int_{-1}^{1} [B]^T[D][B] \det [J(\eta,\zeta)]d\eta d\zeta \qquad (k)$$

where $\det [J(\eta,\zeta)] = \frac{1}{4} AB$ and the indicated integration is performed in the non-dimensionalized coordinate system. Performing this integration, the elemental stiffness matrix results (Figure 7).

The equation of static equilibrium for the element is written:

$$\{Q\}^e = [k]^e\{\delta\}^e \qquad (\ell)$$

where $\{Q\}^e$ is the vector of forces acting at the nodes of the element; $[k]^e$ is the elemental stiffness matrix; and $\{\delta\}^e$ is the vector of elemental displacements,

$$\{\delta\} = \begin{Bmatrix} \delta_i \\ \delta_j \\ \delta_k \\ \delta_\ell \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \\ u_\ell \\ v_\ell \end{Bmatrix} \qquad (m)$$

$$
\begin{bmatrix}
\frac{1}{3}c_{11}\frac{B}{A}+\frac{1}{2}c_{13}+\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{3}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}+\frac{1}{4}c_{33}+\frac{1}{3}c_{23}\frac{A}{B} & -\frac{1}{3}c_{11}\frac{B}{A}+\frac{1}{6}c_{33}\frac{A}{B} & -\frac{1}{3}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}-\frac{1}{4}c_{33}+\frac{1}{6}c_{23}\frac{A}{B} & -\frac{1}{6}c_{11}\frac{B}{A}-\frac{1}{2}c_{13}-\frac{1}{6}c_{33}\frac{A}{B} & -\frac{1}{6}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}-\frac{1}{4}c_{33}-\frac{1}{6}c_{23}\frac{A}{B} & \frac{1}{6}c_{11}\frac{B}{A}-\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{6}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}+\frac{1}{4}c_{33}-\frac{1}{3}c_{23}\frac{A}{B} \\
 & \frac{1}{3}c_{33}\frac{B}{A}+\frac{1}{2}c_{23}+\frac{1}{3}c_{22}\frac{A}{B} & -\frac{1}{3}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}+\frac{1}{4}c_{33}+\frac{1}{6}c_{23}\frac{A}{B} & -\frac{1}{3}c_{33}\frac{B}{A}+\frac{1}{6}c_{22}\frac{A}{B} & -\frac{1}{6}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}-\frac{1}{4}c_{33}-\frac{1}{6}c_{23}\frac{A}{B} & -\frac{1}{6}c_{33}\frac{B}{A}-\frac{1}{2}c_{23}-\frac{1}{6}c_{22}\frac{A}{B} & \frac{1}{6}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}-\frac{1}{4}c_{33}-\frac{1}{3}c_{23}\frac{A}{B} & \frac{1}{6}c_{33}\frac{B}{A}-\frac{1}{3}c_{22}\frac{A}{B} \\
 & & \frac{1}{3}c_{11}\frac{B}{A}-\frac{1}{2}c_{13}+\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{3}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}-\frac{1}{4}c_{33}+\frac{1}{3}c_{23}\frac{A}{B} & \frac{1}{6}c_{11}\frac{B}{A}-\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{6}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}-\frac{1}{4}c_{33}-\frac{1}{3}c_{23}\frac{A}{B} & -\frac{1}{6}c_{11}\frac{B}{A}+\frac{1}{2}c_{13}-\frac{1}{6}c_{33}\frac{A}{B} & -\frac{1}{6}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}+\frac{1}{4}c_{33}-\frac{1}{6}c_{23}\frac{A}{B} \\
 & & & \frac{1}{3}c_{33}\frac{B}{A}-\frac{1}{2}c_{23}+\frac{1}{3}c_{22}\frac{A}{B} & \frac{1}{6}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}+\frac{1}{4}c_{33}-\frac{1}{3}c_{23}\frac{A}{B} & \frac{1}{6}c_{33}\frac{B}{A}-\frac{1}{3}c_{22}\frac{A}{B} & -\frac{1}{6}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}+\frac{1}{4}c_{33}-\frac{1}{6}c_{23}\frac{A}{B} & -\frac{1}{6}c_{33}\frac{B}{A}+\frac{1}{2}c_{23}-\frac{1}{6}c_{22}\frac{A}{B} \\
 & & & & \frac{1}{3}c_{11}\frac{B}{A}+\frac{1}{2}c_{13}+\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{3}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}+\frac{1}{4}c_{33}+\frac{1}{3}c_{23}\frac{A}{B} & -\frac{1}{3}c_{11}\frac{B}{A}+\frac{1}{6}c_{33}\frac{A}{B} & -\frac{1}{3}c_{13}\frac{B}{A}+\frac{1}{4}c_{12}-\frac{1}{4}c_{33}+\frac{1}{6}c_{23}\frac{A}{B} \\
 & & & & & \frac{1}{3}c_{33}\frac{B}{A}+\frac{1}{2}c_{23}+\frac{1}{3}c_{22}\frac{A}{B} & -\frac{1}{3}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}+\frac{1}{4}c_{33}+\frac{1}{6}c_{23}\frac{A}{B} & -\frac{1}{3}c_{33}\frac{B}{A}+\frac{1}{6}c_{22}\frac{A}{B} \\
 & & & & & & \frac{1}{3}c_{11}\frac{B}{A}-\frac{1}{2}c_{13}+\frac{1}{3}c_{33}\frac{A}{B} & \frac{1}{3}c_{13}\frac{B}{A}-\frac{1}{4}c_{12}-\frac{1}{4}c_{33}+\frac{1}{3}c_{23}\frac{A}{B} \\
\text{(SYMMETRIC)} & & & & & & & \frac{1}{3}c_{33}\frac{B}{A}-\frac{1}{2}c_{23}+\frac{1}{3}c_{22}\frac{A}{B}
\end{bmatrix}
$$

FIGURE 7: ELEMENTAL STIFFNESS MATRIX $[k]^e$, FOR A 4-NODED RECTANGULAR FINITE ELEMENT. CLOSED-FORM SOLUTION IN TERMS OF ELEMENT DIMENSIONS (A,B) AND THE VECTOR OF LINEAR, ANISOTROPIC MATERIAL CONSTANTS $(C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33})$ MATRIX $[k]^e$ IS SYMMETRIC.

Performing the multiplication indicated by equation (ℓ) and factoring the six material constants:

$$\{Q\}^e = [k^*]^e\{C\} \qquad (n)$$

where $[k^*]^e$ is the modified elemental stiffness matrix, in terms of (m), (see Figure 8), and $\{C\}$ is the vector of material constants of equation (h).

$$
\left[\begin{array}{cccccc}
\frac{1}{3}\frac{B}{A}(u_i-u_j-\frac{1}{2}u_k+\frac{1}{2}u_\ell) & \frac{1}{4}(v_i+v_j-v_k-v_\ell) & \begin{array}{c}\frac{1}{2}(u_i-u_k)\\ +\frac{1}{3}\frac{B}{A}(v_i-v_j-\frac{1}{2}v_k+\frac{1}{2}v_\ell)\end{array} & 0 & \frac{1}{3}\frac{A}{B}(v_i+\frac{1}{2}v_j-\frac{1}{2}v_k-v_\ell) & \begin{array}{c}\frac{1}{4}(v_i-v_j-v_k+v_\ell)\\ +\frac{1}{3}\frac{A}{B}(u_i+\frac{1}{2}u_j-\frac{1}{2}u_k-u_\ell)\end{array} \\
0 & \frac{1}{4}(u_i-u_j-u_k+u_\ell) & \frac{1}{3}\frac{B}{A}(u_i-u_j-\frac{1}{2}u_k+\frac{1}{2}u_\ell) & \frac{1}{3}\frac{A}{B}(v_i+\frac{1}{2}v_j-\frac{1}{2}v_k-v_\ell) & \begin{array}{c}\frac{1}{2}(v_i-v_k)\\ +\frac{1}{3}\frac{A}{B}(u_i+\frac{1}{2}u_j-\frac{1}{2}u_k-u_\ell)\end{array} & \begin{array}{c}\frac{1}{4}(u_i+u_j-u_k-u_\ell)\\ +\frac{1}{3}\frac{B}{A}(v_i-v_j-\frac{1}{2}v_k+\frac{1}{2}v_\ell)\end{array} \\
\frac{1}{3}\frac{B}{A}(-u_i+u_j+\frac{1}{2}u_k-\frac{1}{2}u_\ell) & \frac{1}{4}(-v_i-v_j+v_k+v_\ell) & \begin{array}{c}\frac{1}{2}(-u_j+u_\ell)\\ +\frac{1}{3}\frac{B}{A}(-v_i+v_j+\frac{1}{2}v_k-\frac{1}{2}v_\ell)\end{array} & 0 & \frac{1}{3}\frac{A}{B}(\frac{1}{2}v_i+v_j-v_k-\frac{1}{2}v_\ell) & \begin{array}{c}\frac{1}{4}(v_i-v_j-v_k+v_\ell)\\ +\frac{1}{3}\frac{A}{B}(\frac{1}{2}u_i+u_j-u_k-\frac{1}{2}u_\ell)\end{array} \\
0 & \frac{1}{4}(u_i-u_j-u_k+u_\ell) & \frac{1}{3}\frac{B}{A}(-u_i+u_j+\frac{1}{2}u_k-\frac{1}{2}u_\ell) & \frac{1}{3}\frac{A}{B}(\frac{1}{2}v_i-v_j-v_k-\frac{1}{2}v_\ell) & \begin{array}{c}\frac{1}{2}(-v_j+v_\ell)\\ +\frac{1}{3}\frac{A}{B}(\frac{1}{2}u_i+u_j-u_k-\frac{1}{2}u_\ell)\end{array} & \begin{array}{c}\frac{1}{4}(-u_i-u_j+u_k+u_\ell)\\ +\frac{1}{3}\frac{B}{A}(-v_i+v_j+\frac{1}{2}v_k-\frac{1}{2}v_\ell)\end{array} \\
\frac{1}{3}\frac{B}{A}(-\frac{1}{2}u_i+\frac{1}{2}u_j+u_k-u_\ell) & \frac{1}{4}(-v_i-v_j+v_k+v_\ell) & \begin{array}{c}\frac{1}{2}(-u_i+u_k)\\ +\frac{1}{3}\frac{B}{A}(-\frac{1}{2}v_i+\frac{1}{2}v_j+v_k-v_\ell)\end{array} & 0 & \frac{1}{3}\frac{A}{B}(-\frac{1}{2}v_i-v_j+v_k+\frac{1}{2}v_\ell) & \begin{array}{c}\frac{1}{4}(-v_i+v_j+v_k-v_\ell)\\ +\frac{1}{3}\frac{A}{B}(-\frac{1}{2}u_i-u_j+u_k+\frac{1}{2}u_\ell)\end{array} \\
0 & \frac{1}{4}(-u_i+u_j+u_k-u_\ell) & \frac{1}{3}\frac{B}{A}(-\frac{1}{2}u_i+\frac{1}{2}u_j+u_k-u_\ell) & \frac{1}{3}\frac{A}{B}(-\frac{1}{2}v_i-v_j+v_k+\frac{1}{2}v_\ell) & \begin{array}{c}\frac{1}{2}(-v_i+v_k)\\ +\frac{1}{3}\frac{A}{B}(-\frac{1}{2}u_i-u_j+u_k+\frac{1}{2}u_\ell)\end{array} & \begin{array}{c}\frac{1}{4}(-u_i-u_j+u_k+u_\ell)\\ +\frac{1}{3}\frac{B}{A}(-\frac{1}{2}v_i+\frac{1}{2}v_j+v_k-v_\ell)\end{array} \\
\frac{1}{3}\frac{B}{A}(\frac{1}{2}u_i-\frac{1}{2}u_j-u_k+u_\ell) & \frac{1}{4}(v_i+v_j-v_k-v_\ell) & \begin{array}{c}\frac{1}{2}(u_j-u_\ell)\\ +\frac{1}{3}\frac{B}{A}(\frac{1}{2}v_i-\frac{1}{2}v_j-v_k+v_\ell)\end{array} & 0 & \frac{1}{3}\frac{A}{B}(-v_i-\frac{1}{2}v_j+\frac{1}{2}v_k+v_\ell) & \begin{array}{c}\frac{1}{4}(-v_i+v_j+v_k-v_\ell)\\ +\frac{1}{3}\frac{A}{B}(-u_i-\frac{1}{2}u_j+\frac{1}{2}u_k+u_\ell)\end{array} \\
0 & \frac{1}{4}(-u_i+u_j+u_k-u_\ell) & \frac{1}{3}\frac{B}{A}(\frac{1}{2}u_i-\frac{1}{2}u_j-u_k+u_\ell) & \frac{1}{3}\frac{A}{B}(-v_i-\frac{1}{2}v_j+\frac{1}{2}v_k+v_\ell) & \begin{array}{c}\frac{1}{2}(v_j-v_\ell)\\ +\frac{1}{3}\frac{A}{B}(-u_i-\frac{1}{2}u_j+\frac{1}{2}u_k+u_\ell)\end{array} & \begin{array}{c}\frac{1}{4}(u_i+u_j-u_k-u_\ell)\\ +\frac{1}{3}\frac{B}{A}(\frac{1}{2}v_i-\frac{1}{2}v_j-v_k+v_\ell)\end{array}
\end{array}\right]
$$

FIGURE 8 : $[K^*]^e$, MODIFIED ELEMENTAL STIFFNESS MATRIX FOR A 4-NODED RECTANGULAR FINITE ELEMENT, CLOSED-FORM SOLUTION IN TERMS OF ELEMENT DIMENSIONS (A,B) AND NODAL DISPLACEMENTS (u,v).

APPENDIX B

COMPUTER PROGRAM DPLISOP

```
                              BRIAN A. EDWARDS
                           LIEUTENANT, U. S. NAVY

          CHARACTERIZATION OF ELASTIC SOLIDS USING FINITE ELEMENT METHODS

                         U. S. NAVAL POSTGRADUATE SCHOOL
                              MONTEREY, CALIFORNIA
                                 DECEMBER 1972

C    *************************************************************************
C    THIS PROGRAM PERFORMS A PLANE STRESS OR A PLANE STRAIN ANALYSIS
C    BY THE FINITE ELEMENT METHOD. NUMERICALLY INTEGRATED ISOPARAMETRIC
C    ELEMENTS ARE USED THROUGHOUT.
C    PROFESSOR GILLES CANTIN,  MARCH 1972
C    *************************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/SOL/BGK(312,64),ALOAD(312),DISP(312),ABGN
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES
     1,NGP,LM(22),LJT(22)
      COMMON/FLPL/COARD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T
C
C   INPUT CARDS NEEDED
C
C   TITLE                     (20A4)          ONE CARD PER PROBLEM
C   PROBLEM PARAMETERS    (7I5,10X,1G24.16)  ONE CARD PER PROBLEM
C    COL.   VARIABLE
C     1-5        NEL   NUMBER OF ELEMENTS (MAX. IS 100)
C     6-10       NJT   NUMBER OF JOINTS (MAX. IS 156)
C    11-15       NMAT   NUMBER OF MATERIALS (MAX. IS 10)
C    16-20       NCLOAD   NUMBER OF CONCENTRATED LOADS
C    21-25       NPBC   NUMBER OF JOINTS WITH BOUNDARY CONDITIONS
C    26-30       NSTRESS   ZERO FOR PLANE STRESS PROBLEMS, ONE FOR P.STRAIN
C    31-35       NGP   NUMBER OF GAUSS POINTS DESIRED IN THE INTEGRATION
C    46-70       T     SPECIMEN THICKNESS:  T=0.0 OR BLANK IS SAME AS
C      UNIT THICKNESS (T=1.0)                      (1G24.16)
C
C   JOINT CARDS  (CARD ONE:1I10,2G31.16 CARD TWO:2G31.16)   NJT PER PROB.
C      TWO CARDS PER JOINT ARE REQUIRED:
C       CARD ONE--JOINT IDENTIFICATION AND COORDINATES
C     1-10       JOINT NUMBER
C    11-41       X  COORDINATE OF THE JOINT
C    42-72       Y  COORDINATE OF THE JOINT
C       CARD TWO--LOADING AT THE JOINT
C     1-31       X COMPONENT OF THE LOAD AT THE JOINT
C    32-62       Y COMPONENT OF THE LOAD AT THE JOINT
C
```

```
C   MATERIAL PROPERTIES      (1I10,2G25.16)  NMAT CARDS PER PROBLEM
C    COL.    VARIABLE
C     1-10       MATERIAL NUMBER
C    11-35       YOUNG'S MODULUS
C    36-60       POISSON'S RATIO
      DIMENSION REACT(157,2),NCODE(156)
      EQUIVALENCE (BGK(1,1),REACT(1,1))
C
C   ELEMENT CARDS                (15I4)           NEL  CARDS PER PROBLEM
C    COL.    VARIABLE
C     1-4        ELEMENT IDENTIFICATION NUMBER
C     5-52       ELEMENT CONNECTIVITY (COUNTERCLOCKWISE)
C    53-56       KIND  1  FOR FOUR NODAL POINT ELEMENT
C                      2  FOR EIGHT NODAL POINT ELEMENT
C                      3  FOR TWELVE NODAL POINT ELEMENT
C    57-60       TYPE  THIS IS THE MATERIAL IDENTIFICATION NUMBER
C
C   BOUNDARY CONDITION CARDS  (2I5)           NPBC  CARDS PER PROBLEM
C    COL.    VARIABLE
C     1-5  JOINT NUMBER
C     6-10 CODE  1 FOR A ROLLER ON AN X AXIS, DISP. IN Y DIRECT. IS ZERO
C                2 FOR A ROLLER ON AN Y AXIS, DISP. IN X DIRECT. IS ZERO
C                3 FOR A FIXED JOINT, BOTH DISP. COMPONENTS ARE ZERO
C                4 FOR A ROLLER ON AN X AXIS, THE DISP. IN THE Y DIRECT.
C              IS SPECIFIED ON THE SECOND JOINT CARD IN COLS. 32-62
C                5 FOR A ROLLER ON A Y AXIS, THE DISP. IN THE X DIRECT.
C              IS SPECIFIED ON THE SECOND JOINT CARD IN COLS. 1-31
C                6 FIXED JOINT WITH DISP. SPECIFIED IN BOTH X AND Y
C              DIRECTIONS AND PUNCHED IN COLS. 1-31 AND COLS. 32-62 OF
C              THE SECOND JOINT CARD (JOINT LOADING)
C                7 FOR A ROLLER ON AN AXIS INCLINED ALPHA DEGREES
C                  WITH RESPECT TO THE X AXIS. THE JOINT CARD MAY
C                  CONTAIN A TANGENTIAL LOAD IN COL.  1-31 AND THE
C                  ANGLE ALPHA DEGREES IN COL. 32-62, SECOND JOINT CARD
C   AS MANY PROBLEMS AS ONE DESIRES MAY BE STACKED, BUT THE LAST CARD OF
C   A RUN MUST CONTAIN THE WORD    STOP    IN  THE FIRST FOUR COLUMNS
      PI=3.14159265358979300
   10 NBAND=0
      DO 20 I=1,156
   20 NCODE(I)=0
      CALL INPUT
      NEQ=2*NJT
      NPEL=NCON(1,13)*4
      DO 300 I=1,NEL
      DO 100 J=1,NPEL
  100 LM(J)=NCON(I,J)
      NPELM=NPEL-1
      DO 200 K=1,NPELM
```

```
      JK=K+1
      DO 200 L=JK,NPEL
  200 NBAND=MAX0(NBAND,IABS(LM(K)-LM(L)))
  300 CONTINUE
      NBAND=(NBAND+1)*2
      DO 350 I=1,NEQ
      ALOAD(I)=0.0D0
      DO 350 J=1,NBAND
  350 BGK(I,J)=0.0D0
      CALL MERGE(NPEL,NEQ)
      CALL BCOND(NBAND)
      CALL CHOL(NEQ,NBAND)
      DO 500 I=1,NPBC
      IJT=NBC(I,1)
  500 NCODE(IJT)=NBC(I,2)
      DO 600 I=1,NJT
      II=I*2-1
      KODE=NCODE(I)
      IF(KODE.EQ.7) GO TO 560
      IF((KODE.EQ.5).OR.(KODE.EQ.6)) GO TO 550
      ALOAD(II)=CLOAD(I,1)+ALOAD(II)
  550 II=II+1
      IF((KODE.EQ.4).OR.(KODE.EQ.6)) GO TO 600
      ALOAD(II)=CLOAD(I,2)+ALOAD(II)
      GO TO 600
  560 ALF=CLOAD(I,2)*PI/180.0D0
      COSA=DCOS(ALF)
      SINA=DSIN(ALF)
      IIP=II+1
      A1=ALOAD(II)*COSA+ALOAD(IIP)*SINA+CLOAD(I,1)
      A2=ALOAD(IIP)*COSA-ALOAD(II)*SINA
      ALOAD(II)=A1
      ALOAD(IIP)=A2
  600 CONTINUE
      CALL SOLV(NEQ,NBAND)
  610 DO 630 I=1,NPBC
      DO 630 J=1,2
  630 REACT(I,J)=0.0D0
      CALL DISPL
      WRITE(6,1000)
 1000 FORMAT(//,' DISPLACEMENTS',//,3X,' JOINT',10X,'  U',22X,'  V',/)
      DO 700 I=1,NJT
      II=I*2-1
      III=II+1
  700 WRITE(6,2000) I,DISP(II),DISP(III)
 2000 FORMAT(2X,I3,2G25.16)
      WRITE(6,3000)
 3000 FORMAT(//,'     REACTIONS',//,3X,' JOINT',10X,'  RX',22X,'  RY',/)
```

```
      WRITE(6,2500) REACT(157,1),REACT(157,2)
 2500 FORMAT(5X,2G25.16)
      CALL PUNCH(NJT,DISP)
      CALL STRESS(NPEL)
  890 GO TO 10
      END


      SUBROUTINE INPUT
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES
     1,NGP,LM(22),LJT(22)
      COMMON/FLPL/COARD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T
      DATA CHK /'STOP'/
      READ (5,1000) TITLE
 1000 FORMAT (20A4)
      IF (TITLE(1).EQ.CHK) STOP
      DO 1050 I=1,156
      DO 1050 J=1,2
      NBC(I,J)=0
      COARD(I,J)=0.0D0
 1050 CLOAD(I,J)=0.0D0
      DO 1060 I=1,10
      DO 1060 J=1,2
 1060 ELCON(I,J)=0.0D0
      DO 1070 I=1,156
      DO 1070 J=1,14
 1070 NCON(I,J)=0
      WRITE(6,1080) TITLE
 1080 FORMAT('1',20A4)
C
C     READ STRUCTURE PARAMETERS
C
      READ (5,1100) NEL,NJT,NMAT,NCLOAD,NPBC,NSTRES,NGP,T
 1100 FORMAT (7I5,10X,1G24.16)
      IF (T.GT.0.0) GO TO 1150
      T=1.0D0
 1150 WRITE (6,1200) NEL,NJT,NMAT,NCLOAD,NPBC,NSTRES,NGP,T
 1200 FORMAT (//,' NUMBER OF ELEMENTS=',I5,//,' TOTAL NUMBER OF JOINTS='
     1,I5,//,' NUMBER OF MATERIALS=',I5,//,' NUMBER OF CONCENTRATED LOAD
     2S=',I5,//,' NUMBER OF JOINTS WITH BOUNDARY CONDITIONS=',I5,//,' TH
     3E VALUE OF NSTRES=',I5,//,' NUMBER OF GAUSS POINTS USED ',I5,
     4//,' ','SPECIMEN IS OF THICKNESS',1G24.16)
C
```

```
      NPBCM=NPBC-1
      DO 4000  K=1,NPBCT
      DO 4000  I=1,NPBCM
      IF(NBC(I,2).EQ.7) GO TO 3600
      GO TO 4000
 3600 N1=NBC(I,1)
      DO 3800  J=I,NPBCM
      NBC(J,1)=NBC(J+1,1)
 3800 NBC(J,2)=NBC(J+1,2)
      NBC(NPBC,1)=N1
      NBC(NPBC,2)=7
 4000 CONTINUE
      RETURN
      END


      SUBROUTINE MERGE(NPEL,NEQ)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION           STK(24,24),AK(24,24),B(3,24)
      COMMON/SOL/BGK(312,64),ALOAD(312),DISP(312),ABGN
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES
     1,NGP,LM(22),LJT(22)
      COMMON/FLPL/COARD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T
      COMMON COORD(12,2),ELAST(3,3),SS(36,24),SN(36,24)
      REWIND 10
      DO 100  I=1,36
      DO 100  J=1,24
      SS(I,J)=0.0D0
  100 SN(I,J)=0.0D0
      DO 200  IK=1,NEL
      DO 110  NN=1,NPEL
      LJT(NN)=NCON(IK,NN)
  110 LM(NN)=(NCON(IK,NN)-1)*2
      NS=3*NPEL
      DO 120  I1=1,NPEL
      I2=LJT(I1)
      DO 120  J1=1,2
  120 COORD(I1,J1)=COARD(I2,J1)
      MATN=NCON(IK,14)
      N=NCON(IK,13)*8
      YM=ELCON(MATN,1)
      PRTIO=ELCON(MATN,2)
      CALL ELASM(ELAST,YM,PRTIO,NSTRES,T)
      CALL QUAD5(STK,AK,B,NS,N,NGP)
      WRITE(10) SS,SN
      DO 200  I=1,NPEL
      DO 200  J=1,NPEL
      DO 200  K=1,2
```

```
      II=LM(I)+K
      KK=2*(I-1)+K
      DO 200 L=1,2
      JJ=LM(J)+L-II+1
      IF(JJ.LE.0) GO TO 200
      LL=2*(J-1)+L
      BGK(II,JJ)=BGK(II,JJ)+STK(KK,LL)
  200 CONTINUE
      SUM=0.0D0
      DO 300 I=1,NEQ
  300 SUM=SUM+BGK(I,1)
      ABGN=SUM*1.0D20
      RETURN
      END

      SUBROUTINE ELASM(D,E,PR,NN,TH)
      IMPLICIT REAL*8 (A-H,O-Z)
C     THIS SUBROUTINE ASSEMBLYS THE ELASTIC MATRIX FOR PLANE
C     STRAIN (NN=1) OR PLANE STRESS (NN=0)
      DIMENSION D(3,3)
      DO 20 I=1,3
      DO 20 J=1,3
   20 D(I,J)=0.0D0
      IF(NN.EQ.0) GO TO 10
C     FORM ELASTIC MATRIX FOR PLANE STRAIN
      ER=TH*E/((1.0D0+PR)*(1.0D0-2.0D0*PR))
      D(1,1)=ER*(1.0D0-PR)
      D(1,2)=ER*PR
      D(2,1)=D(1,2)
      D(2,2)=D(1,1)
      D(3,3)=ER*(0.5D0-PR)
      RETURN
C     FORM ELASTIC MATRIX FOR PLANE STRESS
   10 ER=TH*E/(1.0D0-PR*PR)
      D(1,1)=ER
      D(1,2)=ER*PR
      D(2,1)=D(1,2)
      D(2,2)=D(1,1)
      D(3,3)=ER*(1.0D0-PR)/2.0D0
      RETURN
      END
```

```
      SUBROUTINE QUAD5(STK,AK,B,NS,N,NGP)
      IMPLICIT REAL*8 (A-H,O-Z)
C
C    QUAD5 IS A NUMERICAL INTEGRATION SUBROUTINE THAT FORMS THE STIFFNESS
C    MATRIX FOR ANY OF THE THREE QUADRILATERAL ELEMENTS IN THIS FAMILY.
C    GAUSSIAN QUADRATURE WITH A MAXIMUM OF FIVE ORDINATES IS USED.
C
C    THE SUBROUTINE IS CALLED WITH THE FOLLOWING STATEMENT:
C    CALL QUAD5(STK,AK,SS,SN,NS,N,NGP)
C    THE CALLING PROGRAM MUST CONTAIN THE FOLLOWING COMMON STATEMENT:
C    COMMON COORD(12,2),ELAST(3,3)
C        STK   IS AN ARRAY DIMENSIONED NXN THAT WILL CONTAIN THE STIFFNES
C              MATRIX UPON RETURN TO THE CALLING PROGRAM
C        SS AND SN ARE THE STRESS AND STRAIN TRANSFORMATION MATRICES,
C              VERSES NODAL POINT DISPLACEMENTS; BOTH ARE DIMENSIONED
C              NSXN--NS IS 3*NPT, N IS 2*NPT. SCRATCH ACCESS ON UNIT 10.
C        AK    IS AN ARRAY DIMENSIONED NXN REQUIRED IN FORMK
C        B     IS AN ARRAY DIMENSIONED 2XN REQUIRED IN FORMK
C
C        NGP IS THE NUMBER OF GAUSS POINTS UED IN THE INTEGRATION
C
C        COORD(12,2)  IS AN ARRAY CONTAINING THE COORDINATES OF ALL
C                     THE NODAL POINTS OF THE ELEMENT TO BE CONSTRUCTED
C        ELAST(3,3)   IS AN ARRAY CONTAINING ALL THE ELASTIC CONSTANTS
C                     OF THE ELEMENT, THE ORDER OF THE STRESS COMPONENTS
C                     IS: SIGMAX,SIGMAY,TAUXY
C
      DIMENSION AK(24,24),STK(24,24),B(3,24)
      DIMENSION XI(5),AI(5),AIA(5,5)
      DIMENSION X2(2),X3(3),X4(4),X5(5),A2(2),A3(3),A4(4),A5(5)
      DIMENSION XYL(4,2),XYQ(8,2),XYC(12,2)
C
C               ETA                        ETA                        ETA
C                +                          +                          +
C         U2----+------U1           U3----U2----U1           U4--U3+-U2--U1
C         |      +      |           |      +     |           |      +     |
C         |      +      |           |      +     |           |      +     |
C         |      +      |           |      +     |           |      +     |
C         |      +      |           |      +     |           U5     +     U12
C         |      +      |           |      +     |           |      +     |
C         |      ++++++++XI         U4     ++++++U8++XI      |      ++++++++XI
C         |             |           |            |           |            |
C         |             |           |            |           U6           U11
C         |             |           |            |           |            |
C         |             |           |            |           |            |
C         U3-----------U4           U5----U6----U7           U7--U8--U9--U10
C
```

```
C     CODED BY PROFESSOR GILLES CANTIN AT THE NAVAL POSTGRADUATE SCHOOL
C
      COMMON COORD(12,2),ELAST(3,3),SS(36,24),SN(36,24)
      DATA X2/0.577350269189626D0,-0.577350269189626D0/
      DATA A2/1.0D0,1.0D0/
      DATA X3/0.774596669241483D0,0.0D0,-0.774596669241483D0/
      DATA A3/0.555555555555556D0,0.888888888888889D0,0.5555555555555
     156D0/
      DATA X4/0.861136311594053D0,0.339981043584856D0,-0.339981043584856
     1D0,-0.861136311594053/
      DATA A4/0.347854845137454D0,0.652145154862546D0,0.652145154862546D
     10,0.347854845137454D0/
      DATA X5/0.906179845938664D0,0.538469310105683D0,0.0D0,-0.553846931
     10105683D0,-0.906179845938664D0/
      DATA A5/0.236926885056189D0,0.478628670499366D0,0.568888888888888
     1D0,0.478628670499366D0,0.236926885056189D0/
      DATA XYL/1.0D0,-1.0D0,-1.0D0,1.0D0,1.0D0,1.0D0,-1.0D0,-1.0D0/
      DATA XYQ/1.0D0,0.0D0,-1.0D0,-1.0D0,-1.0D0,0.0D0,1.0D0,1.0D0,1.0D0,
     11.0D0,1.0D0,0.0D0,-1.0D0,-1.0D0,-1.0D0,0.0D0/
      DATA XYC/1.0D0,0.333333333333333D0,-0.333333333333333D0,-1.0D0,
     1 -1.0D0,-1.0D0,-1.0D0,-0.333333333333333D0,0.333333333333333D0,
     11.0D0,1.0D0,1.0D0,1.0D0,1.0D0,1.0D0,1.0D0,0.333333333333333D0,
     1-0.333333333333333D0,-1.0D0,-1.0D0,-1.0D0,-1.0D0,-0.333333333333
     1333D0,0.333333333333333D0/
      IF(NGP.EQ.0) NGP=5
      DO 100 I=1,N
      DO 100 J=1,N
  100 STK(I,J)=0.0D0
      JUMP=NGP-1
      DO 140 I=1,NGP
      GO TO (132,133,134,135),JUMP
  132 XI(I)=X2(I)
      AI(I)=A2(I)
      GO TO 140
  133 XI(I)=X3(I)
      AI(I)=A3(I)
      GO TO 140
  134 XI(I)=X4(I)
      AI(I)=A4(I)
      GO TO 140
  135 XI(I)=X5(I)
      AI(I)=A5(I)
  140 CONTINUE
      DO 150 I=1,NGP
      DO 150 J=1,NGP
  150 AIA(I,J)=AI(I)*AI(J)
      DO 200 I=1,NGP
      X=XI(I)
```

```
      DO 200 J=1,NGP
      Y=XI(J)
      CALL FORMK(AK,X,Y,B,N)
      DO 200 K=1,N
      DO 200 L=1,N
  200 STK(K,L)=STK(K,L)+AIA(I,J)*AK(K,L)
      DO 300 I=1,N
      DO 300 J=I,N
      STK(I,J)=(STK(I,J)+STK(J,I))*0.5D0
  300 STK(J,I)=STK(I,J)
      NPT=N/2
      IGO=NPT/4
      DO 1000 I=1,NPT
      GO TO (400,500,600),IGO
  400 X=XYL(I,1)
      Y=XYL(I,2)
      GO TO 700
  500 X=XYQ(I,1)
      Y=XYQ(I,2)
      GO TO 700
  600 X=XYC(I,1)
      Y=XYC(I,2)
  700 CALL FORMK(AK,X,Y,B,N)
      DO 800 L=1,3
      DO 800 M=1,N
      AK(L,M)=0.0D0
      DO 800 NN=1,3
  800 AK(L,M)=AK(L,M)+ELAST(L,NN)*B(NN,M)
      II=3*(I-1)
      DO 900 J=1,3
      IJ=II+J
      DO 900 K=1,N
      SS(IJ,K)=AK(J,K)
  900 SN(IJ,K)=B(J,K)
 1000 CONTINUE
      RETURN
      END

      SUBROUTINE FORMK(AK,X,Y,B,N)
      IMPLICIT REAL*8 (A-H,O-Z)
C
C
C THIS SUBROUTINE FORMS THE STIFFNESS AND THE B MATRIX AS FUNCTIONS OF
C XI AND ETA FOR THE THREE DIFFERENT QUADRILATERAL ELEM. IN THIS FAMILY
C
C
      DIMENSION AJ(2,2),AJIN(2,2),DNX(2,12),W1(2,12),B(3,24)
```

```
     1,B1(24,3),AK(24,24)
      COMMON COORD(12,2),ELAST(3,3),SS(36,24),SN(36,24)
C INITIALIZE
      DO 10 I=1,3
      DO 10 J=1,N
   10 B(I,J)=0.0D0
      ZERO=0.0D0
      ONE=1.0D0
      TWO=2.0D0
      FOUR=4.0D0
      NPT=N/2
C
C FORM (2XNPT) MATRIX OF DERIV. OF THE INTERP. FUNCT. WRT XI AND ETA
C
      IGO=N/8
      GO TO (20,30,40),IGO
C
C     LINEAR FUNCTIONS
C
   20 W1(1,3)=-(ONE-Y)/FOUR
      W1(1,4)=-W1(1,3)
      W1(1,1)= (ONE+Y)/FOUR
      W1(1,2)=-W1(1,1)
      W1(2,3)=-(ONE-X)/FOUR
      W1(2,4)=-(ONE+X)/FOUR
      W1(2,1)=-W1(2,4)
      W1(2,2)=-W1(2,3)
      GO TO 50
   30 CONTINUE
C
C    QUADRATIC FUNCTIONS
C
      TXPY=TWO*X+Y
      TXMY=TWO*X-Y
      TYPX=TWO*Y+X
      TYMX=TWO*Y-X
      OMY=ONE-Y
      OPY=ONE+Y
      OPX=ONE+X
      OMX=ONE-X
      W1(1,5)=OMY*TXPY/FOUR
      W1(1,6)=-OMY*X
      W1(1,7)=OMY*TXMY/FOUR
      W1(I,8)=OPY*OMY/TWO
      W1(1,1)=OPY*TXPY/FOUR
      W1(1,2)=-OPY*X
      W1(1,3)= OPY*TXMY/FOUR
      W1(1,4)=-OPY*OMY/TWO
```

```
      W1(2,5)=OMX*TYPX/FOUR
      W1(2,6)=-OPX*OMX/TWO
      W1(2,7)=OPX*TYMX/FOUR
      W1(2,8)=-Y*OPX
      W1(2,1)=OPX*TYPX/FOUR
      W1(2,2)=OPX*OMX/TWO
      W1(2,3)=OMX*TYMX/FOUR
      W1(2,4)=-Y*OMX
      GO TO 50
   40 CONTINUE
C
C     CUBIC FUNCTIONS
C
      OPX=ONE+X
      OPY=ONE+Y
      OMX=ONE-X
      OMY=ONE-Y
      TPTMNX=3.0D0+2.0D0*X-9.0D0*X*X
      TMTMNX=3.0D0-2.0D0*X-9.0D0*X*X
      TPTMNY=3.0D0+2.0D0*Y-9.0D0*Y*Y
      TMTMNY=3.0D0-2.0D0*Y-9.0D0*Y*Y
      TYPO=3.0D0*Y+ONE
      TYMO=3.0D0*Y-ONE
      TXPO=3.0D0*X+ONE
      TXMO=3.0D0*X-ONE
      TMTPEX=10.0D0-27.0D0*X*X+18.0D0*X-9.0D0*Y*Y
      TMTMEX=10.0D0-27.0D0*X*X-18.0D0*X-9.0D0*Y*Y
      TMNPEY= 10.0D0-9.0D0*X*X+18.0D0*Y-27.0D0*Y*Y
      TMNMEY= 10.0D0-9.0D0*X*X-18.0D0*Y-27.0D0*Y*Y
      TTT=32.0D0
      TTN=32.0D0/9.0D0
      W1(1,7)= OMY*TMTPEX/TTT
      W1(1,8)=-TPTMNX*OMY/TTN
      W1(1,9)=TMTMNX*OMY/TTN
      W1(1,10)=-OMY*TMTMEX/TTT
      W1(1,11)=-OPY*OMY*TYMO/TTN
      W1(1,12)=OPY*OMY*TYPO/TTN
      W1(1,1)=-OPY*TMTMEX/TTT
      W1(1,2)=TMTMNX*OPY/TTN
      W1(1,3)=-TPTMNX*OPY/TTN
      W1(1,4) =OPY*TMTPEX/TTT
      W1(1,5) =-OPY*OMY*TYPO/TTN
      W1(1,6) = OPY*OMY*TYMO/TTN
      W1(2,7) = OMX*TMNPEY/TTT
      W1(2,8)= OPX*OMX*TXMO/TTN
      W1(2,9)=-OPX*OMX*TXPO/TTN
      W1(2,10) = OPX*TMNPEY/TTT
      W1(2,11)=-TPTMNY*OPX/TTN
```

```
      W1(2,12)= TMTMNY*OPX/TTN
      W1(2,1) =-OPX*TMNMEY/TTT
      W1(2,2)=OPX*OMX*TXPO/TTN
      W1(2,3)=-OPX*OMX*TXMO/TTN
      W1(2,4) =-OMX*TMNMEY/TTT
      W1(2,5) =OMX*TMTMNY/TTN
      W1(2,6) =-OMX*TPTMNY/TTN
   50 CONTINUE
C
C FORM JACOBIAN OF THE TRANSFORMATION IN AJ(I,J)
C
      DO 100 I=1,2
      DO 100 J=1,2
      AJ(I,J)=ZERO
      DO 100 K=1,NPT
  100 AJ(I,J)=AJ(I,J)+W1(I,K)*COORD(K,J)
C
C CALCULATE DETERMINANT OF THE JACOBIAN
C
      DTJ=AJ(1,1)*AJ(2,2)-AJ(1,2)*AJ(2,1)
C
C INVERT JACOBIAN IN AJIN
C
      AJIN(1,1)=AJ(2,2)/DTJ
      AJIN(1,2)=-AJ(1,2)/DTJ
      AJIN(2,1)=-AJ(2,1)/DTJ
      AJIN(2,2)=AJ(1,1)/DTJ
C
C FORM (2XNPT) MATRIX OF DERIVATIVES OF INTERP. FUNCT WRT X AND Y
C
      DO 200 I=1,2
      DO 200 J=1,NPT
      DNX(I,J)=ZERO
      DO 200 K=1,2
  200 DNX(I,J)=DNX(I,J)+AJIN(I,K)*W1(K,J)
C
C FORM B(I,J) MATRIX
C
      DO 300 I=1,NPT
      IOD=2*I-1
      IEV=2*I
      B(1,IOD)=DNX(1,I)
      B(2,IEV)=DNX(2,I)
      B(3,IEV)=DNX(1,I)
  300 B(3,IOD)=DNX(2,I)
C
C FORM STIFFNESS BY THE CONGRUENT TRANSFORMATION  BT*D*B
C
```

```
      DO 500 I=1,N
      DO 500 J=1,3
      B1(I,J)=ZERO
      DO 500 K=1,3
  500 B1(I,J)=B1(I,J)+B (K,I)*ELAST(K,J)
      DO 600 I=1,N
      DO 600 J=1,N
      AK(I,J)=ZERO
      DO 600 K=1,3
  600 AK(I,J)=AK(I,J)+B1(I,K)*B(K,J)
      DO 700 I=1,N
      DO 700 J=I,N
      AK(I,J)=((AK(I,J)+AK(J,I))/2.0D0)*DTJ
  700 AK(J,I)=AK(I,J)
      RETURN
      END


      SUBROUTINE CHOL(NEQ,NBAND)
      IMPLICIT REAL*8 (A-H,O-Z)
C***********************************************************************
C    THIS SUBROUTINE DOES THE CHOLESKI DECOMPOSITION OF A MATRIX  (A)
C    WHEN  (A) IS SYMMETIRC AND BANDED,  (A) = (L)*(LT)
C    THE MATRIX (A) MUST BE FURNISHED IN RECTANGULAR FORM A(NEQ,NBAND)
C    WITH ALL ITS DIAGONAL ELEMENTS IN THE FIRST COLUMN
C    NEQ  IS THE TOTAL NUMBER OF EQUATIONS
C    NBAND IS THE HALF-BANDWIDTH OF THE SYSTEM
C     CODED BY PROF. CANTIN, DECEMBER 1971
C***********************************************************************
C
      COMMON/SOL/ A(312,64),B(312),X(312),ABGN
      ZRO=0.0D0
      ANEQ=NEQ
      APZRO=ABGN*1.0D-25/ANEQ
      DO 300 I=1,NEQ
      DIAG=A(I,1)
      IF(DIAG.LE.ZRO) GO TO 400
      IF(DIAG.LE.APZRO) WRITE(6,1000) I
 1000 FORMAT(//5X,' SYSTEM IS NEARLY SINGULAR AT EQUATION ',I5//)
      DIAG=DSQRT(DIAG)
      DO 200 J=1,NBAND
  200 A(I,J)=A(I,J)/DIAG
      DO 260 J=2,NBAND
      L=I+J-1
      IF(L.GT.NEQ) GO TO 260
      AA=A(I,J)
      IF(AA.EQ.ZRO) GO TO 260
      DO 250 K=J,NBAND
```

```
      M=1+K-J
  250 A(L,M)=A(L,M)-AA*A(I,K)
  260 CONTINUE
  300 CONTINUE
      DNM=0.0D0
      DO 350 I=1,NEQ
  350 DNM=DNM+1.0D0/A(I,1)
      ANUM=ABGN*1.0D-20
      CNM=ANUM/DNM
      WRITE(6,2000) ANUM,DNM,CNM
 2000 FORMAT(//,5X,' CONDITION NUMBER = (',1G24.16,')/(',1G24.16,')=',
     11G22.16,//)
      RETURN
  400 WRITE(6,1100) I,DIAG
 1100 FORMAT(//5X,' SYSTEM IS SINGULAR AT EQUATION ',I5,/,5X,' OR INDEFI
     1NITE, DIAG = ',1PG24.16,//)
      STOP
      END


      SUBROUTINE SOLV(NEQ,NBAND)
      IMPLICIT REAL*8 (A-H,O-Z)
C***********************************************************************
C     THIS SUBROUTINE PERFORMS A FORWARD SUBSTITUTION FOLLOWED BY A
C     BACKWARD SUBSTITUTION IN THE SYSTEM  (A)*(X) = (B) WHERE THE MATRIX
C     (A) HAS ALREADY BEEN DECOMPOSED BY THE SUBROUTINE CHOL
C     (A) IS THE MATRIX OF COEEFICIENTS AFTER A CALL TO S.R.CHOL
C     X IS A VECTOR THAT CONTAINS THE ANSWERS UPON EXIT FROM THE S.R.
C     NEQ AND NBAND ARE THE SAME AS IN CHOL
C
C     AFTER ONE CALL TO S.R.CHOL MANY CALLS TO SOLV CAN BE MADE WITH
C     DIFFERENT VALUES FOR THE VECTOR (B) ,(B) IS DESTROYED EVERY TIME
C      CODED BY PROF. CANTIN, DECEMBER 1971
C***********************************************************************
C
      COMMON/SOL/ A(312,64),B(312),X(312),ABGN
      ZRO=0.0D0
      DO 200 I=1,NEQ
      IF(B(I).EQ.ZRO) GO TO 200
      B(I)=B(I)/A(I,1)
      DO 100 J=2,NBAND
      L=I+J-1
      IF(L.GT.NEQ) GO TO 100
      AA=A(I,J)
      B(L)=B(L)-AA*B(I)
  100 CONTINUE
  200 CONTINUE
      X(NEQ)=B(NEQ)/A(NEQ,1)
```

```
      DO 400 L=2,NEQ
      K=NEQ-L+1
      SUM=ZRO
      DO 300 J=2,NBAND
      M=J+K-1
      IF(NEQ.LT.M) GO TO 300
      SUM=SUM+A(K,J)*X(M)
  300 CONTINUE
  400 X(K)=(B(K)-SUM)/A(K,1)
      RETURN
      END


      SUBROUTINE STRESS(NPEL)
      IMPLICIT REAL*8 (A-H,O-Z)
      CCMMON/SOL/BGK(312,64),ALOAD(312),DISP(312),ABGN
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES
     1,NGP,LM(22),LJT(22)
      COMMON/FLPL/COARD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T
      COMMON COORD(12,2),ELAST(3,3),SS(36,24),SN(36,24)
      DIMENSION SSJNT(100,7),SNJNT(100,7),SSEL(36),SNEL(36),DSPEL(24)
      EQUIVALENCE (BGK(1,1),SSJNT(1,1)),(BGK(1,5),SNJNT(1,1)),(BGK(1,9),
     1SSEL(1)),(BGK(1,10),SNEL(1)),(BGK(1,11),DSPEL(1))
      PI=3.141592653589793D0
      REWIND 10
      DO 50 I=1,NJT
      DO 50 J=1,7
      SNJNT(I,J)=0.0D0
   50 SSJNT(I,J)=0.0D0
      DO 300 I=1,NEL
      READ(10) SS,SN
      DO 100 J=1,NPEL
      JA=NCON(I,J)*2-1
      JJ=J*2-1
      DSPEL(JJ)=DISP(JA)
  100 DSPEL(JJ+1)=DISP(JA+1)
      NJLM=NPEL*2
      NPELT=NPEL*3
      DO 200 I1=1,NPELT
      SSEL(I1)=0.0D0
      SNEL(I1)=0.0D0
      DO 200 J1=1,NJLM
      SSEL(I1)=SSEL(I1)+SS(I1,J1)*DSPEL(J1)
  200 SNEL(I1)=SNEL(I1)+SN(I1,J1)*DSPEL(J1)
      IF(NMAT.EQ.1) GO TO 220
      WRITE(6,4000) I
      ILO=0
      IUP=0
```

```
      DO 210 J=1,NPEL
      IJT=NCON(I,J)
      ILO=IUP+1
      IUP=ILO+2
  210 WRITE(6,5000) IJT,(SSEL(K),K=ILO,IUP),(SNEL(L),L=ILO,IUP)
  220 CONTINUE
 5000 FORMAT(5X,I5,6G20.13)
 4000 FORMAT(//,2X,' S-T-R-E-S-S-E-S  /  S-T-R-A-I-N-S  FOR  ELEMENT',I3
     1,        //,5X,' JOINT',5X,' SSX ',17X,' SSY ',17X,'SSXY',17X,'SNX'
     2,17X,' SNY ',17X,' SNXY ',//)
      ICT=0
      DO 300 I2=1,NPEL
      I2A=NCON(I,I2)
      DO 250 J2=1,3
      ICT = ICT + 1
      SSJNT(I2A,J2)=SSEL(ICT)+SSJNT(I2A,J2)
  250 SNJNT(I2A,J2)=SNEL(ICT)+SNJNT(I2A,J2)
  300 SNJNT(I2A,7)=SNJNT(I2A,7)+1.
      DO 500 I3=1,NJT
      DO 400 J3=1,3
      SSJNT(I3,J3)=SSJNT(I3,J3)/SNJNT(I3,7)
  400 SNJNT(I3,J3)=SNJNT(I3,J3)/SNJNT(I3,7)
      GAU=SSJNT(I3,3)
      SXPSY=(SSJNT(I3,1)+SSJNT(I3,2))/2.
      SXMSY=(SSJNT(I3,1)-SSJNT(I3,2))/2.
      AINT=DSQRT(SXMSY*SXMSY+GAU*GAU)
      SSJNT(I3,4)=SXPSY+AINT
      SSJNT(I3,5)=SXPSY-AINT
      SSJNT(I3,6)=(SSJNT(I3,4)-SSJNT(I3,5))/2.0D0
      IF(DABS(SXMSY).LE.1.0D-6) GO TO 420
      RAN=-GAU/SXMSY
      IF(DABS(RAN).LE.1.0D-2) GO TO 450
      RAN=DATAN(RAN)
      GO TO 450
  420 RAN=DSIGN(PI,GAU)/2.0D0
  450 SSJNT(I3,7)=RAN*180.0D0/(2.0D0*PI)
      GAU=SNJNT(I3,3)
      SXPSY=(SNJNT(I3,1)+SNJNT(I3,2))/2.0D0
      SXMSY=(SNJNT(I3,1)-SNJNT(I3,2))/2.0D0
      AINT=DSQRT(SXMSY*SXMSY+GAU*GAU)
      SNJNT(I3,4)=SXPSY+AINT
      SNJNT(I3,5)=SXPSY-AINT
      SNJNT(I3,6)=(SNJNT(I3,4)-SNJNT(I3,5))/2.0D0
  500 SNJNT(I3,7)=SSJNT(I3,7)
      WRITE(6,1000)
 1000 FORMAT(///,1X,' A-V-E-R-A-G-E  S-T-R-E-S-S-E-S  AT  THE  JOINTS',
     1//,67X,'PRINCIPLE STRESSES',11X,'MAX. SHEAR',8X,'PRINCIPLE AXIS ',
     1        //,2X,' JOINT',4X,' SSX  ',12X,'SSY',15X,'SSXY',12X,'SSXM'
```

```
     1,12X,'SSYM',12X,'SSXYM',15X,'TET(DEG.)',//)
      DO 600 I=1,NJT
  600 WRITE(6,2000) I,(SSJNT(I,J),J=1,7)
      WRITE(6,3000)
 3000 FORMAT(//,1X,' A-V-E-R-A-G-E  S-T-R-A-I-N-S  AT  THE  JOINTS',
     1//,67X,'PRINCIPLE  STRAINS',11X,'MAX. SHEAR',8X,'PRINCIPLE AXIS ',
     1        //,2X,' JOINT',4X,' SSX   ',12X,'SSY',15X,'SSXY',12X,'SSXM'
     1,12X,'SSYM',12X,'SSXYM',15X,'TET(DEG.)',//)
      DO 700 I=1,NJT
  700 WRITE(6,2000) I,(SNJNT(I,J),J=1,6),SSJNT(I,7)
 2000 FORMAT(2X,I4,6G17.10,2X,1G19.10)
      RETURN
      END


      SUBROUTINE BCOND(NBAND)
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/FLPL/CJARD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T
      CCMMON/SOL/BGK(312,64),ALOAD(312),DISP(312),ABGN
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES
     1,NGP,LM(22),LJT(22)
      PI=3.141592653589793D0
      NEQ=NJT*2
      DO 200 I=1,NPBC
      IJT=NBC(I,1)
      IEQ=IJT*2-1
      KODE=NBC(I,2)
      IDIA=0
      GO TO (110,120,130,140,150,160,170),KODE
  110 IEQ=IEQ+1
  112 BGK(IEQ,1)=ABGN
      IF(IDIA.EQ.1) GO TO 132
      GO TO 200
  120 GO TO 112
  130 IDIA=1
      GO TO 112
  132 IDIA=0
      GO TO 110
  140 DXY=CLOAD(IJT,2)
  142 IEQ=IEQ+1
  143 ALOAD(IEQ)=ALOAD(IEQ)-BGK(IEQ,1)*DXY
      BGK(IEQ,1)=ABGN
      DO 146 J=2,NBAND
      IROW=IEQ-J+1
      ICOL=IEQ-1+J
      IF(IROW.LT.1) GO TO 144
      ALOAD(IROW)=ALOAD(IROW)-BGK(IROW,J)*DXY
  144 IF(ICOL.GT.NEQ) GO TO 146
```

```
      ALOAD(ICOL)=ALOAD(ICOL)-BGK(IEQ,J)*DXY
  146 CONTINUE
      IF(IDIA.EQ.1) GO TO 162
      GO TO 200
  150 DXY=CLOAD(IJT,1)
      GO TO 143
  160 IDIA=1
      DXY=CLOAD(IJT,1)
      GO TO 143
  162 IDIA=0
      GO TO 140
  170 ALPHA=CLOAD(IJT,2)*PI/180.0D0
      CAL=DCOS(ALPHA)
      SAL=DSIN(ALPHA)
      IEQP=IEQ+1
      DO 180 J=3,NBAND
      IROW=IEQ-J+2
      JM=J-1
      IF(IROW.LT.1) GO TO 175
      A1=BGK(IROW,JM)*CAL+BGK(IROW,J)*SAL
      A2=BGK(IROW,J)*CAL-BGK(IROW,JM)*SAL
      BGK(IROW,JM)=A1
      BGK(IROW,J )=A2
  175 IF((IEQ+J-1).GT.NEQ) GO TO 180
      A3=BGK(IEQ,J)*CAL+BGK(IEQP,JM)*SAL
      A4=BGK(IEQP,JM)*CAL-BGK(IEQ,J)*SAL
      BGK(IEQ ,J )=A3
      BGK(IEQP,JM)=A4
  180 CONTINUE
      A1=BGK(IEQ,1)*CAL*CAL+BGK(IEQ,2)*CAL*SAL*2.0D0+BGK(IEQP,1)*SAL*SAL
      A2=BGK(IEQ,2)*(CAL*CAL-SAL*SAL)+(BGK(IEQP,1)-BGK(IEQ,1))*SAL*CAL
      BGK(IEQ,1)=A1
      BGK(IEQ,2)=A2
      BGK(IEQP,1)=ABGN
  200 CONTINUE
      RETURN
      END

      SUBROUTINE DISPL
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/FLPL/COORD(156,2),CLOAD(156,2),ELCON(10,2),TITLE(20),T
      COMMON/SOL/BGK(312,64),ALOAD(312),DISP(312),ABGN
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES
     1,NGP,LM(22),LJT(22)
      DIMENSION REACT(157,2)
      EQUIVALENCE (BGK(1,1),REACT(1,1))
      PI=3.141592653589793D0
```

```
      DO 200 I=1,NPBC
      IJT=NBC(I,1)
      IEQ=IJT*2-1
      KODE=NBC(I,2)
      IDIA=0
      DPBC=0.0D0
      GO TO (110,120,130,140,150,160,170),KODE
  110 IEQ=IEQ+1
      JR=2
  112 REACT(I,JR)=-DISP(IEQ)*ABGN
      DISP(IEQ)=DPBC
      IF(IDIA.EQ.1) GO TO 132
      IF(IDIA.EQ.2) GO TO 162
      GO TO 200
  120 JR=1
      GO TO 112
  130 IDIA=1
      JR=1
      GO TO 112
  132 IDIA=0
      GO TO 110
  140 DPBC=CLOAD(IJT,2)
      GO TO 110
  150 JR=1
      DPBC=CLOAD(IJT,1)
      GO TO 112
  160 IDIA=2
      JR=1
      DPBC=CLOAD(IJT,1)
      GO TO 112
  162 IDIA=0
      DPBC=CLOAD(IJT,2)
      GO TO 110
  170 ALPHA=CLOAD(IJT,2)*PI/180.0D0
      CAL=DCOS(ALPHA)
      SAL=DSIN(ALPHA)
      A1=DISP(IEQ)*CAL
      A2=DISP(IEQ)*SAL
      IEQP=IEQ+1
      A3=-DISP(IEQP)*SAL*ABGN
      A4= DISP(IEQP)*CAL*ABGN
      DISP(IEQ)=A1
      DISP(IEQP)=A2
      REACT(I   ,1)=-A3
      REACT(I   ,2)=-A4
  200 CONTINUE
      REACT(157,1)=0.0D0
      REACT(157,2)=0.0D0
```

```
      DO 300 I=1,NP3C
      REACT(157,1)=REACT(157,1)+REACT(I,1)
  300 REACT(157,2)=REACT(157,2)+REACT(I,2)
      RETURN
      END


      SUBROUTINE PUNCH
C     *****************************************************************
C
C     SUBROUTINE PUNCH PRODUCES DISPLACEMENT OUTPUT ON PUNCHED CARDS
C     TO BE USED AS DATA INPUT TO PROGRAM ELASTIC
C
C     *****************************************************************
C
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON/SOL/BGK(312,64),ALOAD(312),DISP(312),ABGN
      COMMON/INT/NEL,NJT,NMAT,NCLOAD,NPBC,NCON(100,14),NBC(156,2),NSTRES
     1,NGP,LM(22),LJT(22)
      DO 20 I=1,NJT
      II=I*2-1
      III=II+1
      WRITE(7,10) I,DISP(II),DISP(III)
   10 FORMAT(1I10,2G24.16)
   20 CONTINUE
      RETURN
      END
```

COMPUTER PROGRAM DLASTIC

```
                         BRIAN A. EDWARDS
                       LIEUTENANT, U. S. NAVY

      CHARACTERIZATION OF ELASTIC SOLIDS USING FINITE ELEMENT METHODS

                     U. S. NAVAL POSTGRADUATE SCHOOL
                         MONTEREY, CALIFORNIA
                            DECEMBER 1972

C     ***************************************************************
C
C     PROGRAM DLASTIC DETERMINES THE SIX ELASTIC CONSTANTS <C11,C12,C13,
C     C22,C23,C33>BY DIRECT INVERSION OF THE FINITE ELEMENT FORMULATION
C     IN ORDER TO CHARACTERIZE A GENERAL (LINEAR) ANISOTROPIC MATERIAL.
C     DISPLACEMENTS AND FORCES AT ALL NODAL POINTS ARE KNOWN.
C
      IMPLICIT REAL*8(A-H,O-Z)
C
C     ELEMENT GEOMETRY USED IN DLASTIC IS PRESENTLY RESTRICTED TO THE
C     FOLLOWING 4-NODED, RECTANGULAR SHAPE:
C
C                            J .____________. I
C                              |     V|     |
C                          B   |      |___U |
C                              |            |
C                            K .____________. L
C                                    A
C
C     THE HORIZONTAL DIMENSION, A, IS CALCULATED IN DLASTIC AS THE AVER-
C     AGE OF TOP ELEMENT AND BOTTOM ELEMENT HORIZ. DIMENSIONS.
C     THE VERTICAL DIMENSION, B, IS CALCULATED AS THE AVERAGE OF RIGHT
C     AND LEFT VERTICAL SIDE DIMENSIONS.
C
C     ***************************************************************
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN(
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,
     1TITLE(20),T
C     ***************************************************************
C
C     INPUT CARDS REQUIRED:
C      TITLE (20A4)
C      PARAMETERS (2I10,10X,1G24.16)
C        COL 1-10 NUMBER OF ELEMENTS IN STRUCTURE (NEL)  MAX. NEL=50
C           11-20            JOINTS              (NJT)  MAX. NJT=100
C           31-55 THICKNESS, T.   T=0.0 OR BLANK SAME AS UNIT THICK T=1.
C
```

```
C        ELEMENTAL CONNECTIVITY (1-CARD PER ELEMENT)
C          COL 1-10 ELEMENT ID, NEID (1I10)
C               11-15  JOINT ID, JOINT I OF ELEMENT
C               16-20  JOINT ID, JOINT J OF ELEMENT
C               21-25  JOINT ID, JOINT K OF ELEMENT
C               26-30  JOINT ID, JOINT L OF ELEMENT
C
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO
C
C       JOINT COORDINATES (1-CARD PER JOINT)
C        COL 1-10 JOINT ID, JID (1I10)
C             11-34 X-COORDINATE OF JOINT (1G24.16)
C             35-58 Y-COORDINATE OF JOINT (1G24.16)
C       JOINT FORCES (1-CARD PER JOINT)
C        COL 1-10 JOINT ID, JID (1I10)
C             11-34 RESULTANT HORIZONTAL FORCE ACTING ON JOINT (1G24.16)
C             35-58 RESULTANT VERTICAL   FORCE ACTING ON JOINT (1G24.16)
C       JOINT DISPLACEMENTS (1-CARD PER JOINT)
C        COL 1-10 JOINT ID, JID (1I10)
C             11-34 HORIZONTAL DISPLACEMENT OF JOINT (1G24.16)
C             35-58 VERTICAL   DISPLACEMENT OF JOINT (1G24.16)
C
C              CODED BY BRIAN A. EDWARDS, LT., USN (AUG. 72)
C       ****************************************************************
   99 CALL INPUT
      CALL MERGE
      CALL BMULT
      CALL SINGL
      CALL INVRT(BSYM,6,B,L1,L2)
      CALL ANSER
      GO TO 99
      END


      SUBROUTINE INPUT
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN(
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,
     1TITLE(20),T
      DATA CHK/'STOP'/
      READ (5,10) TITLE
   10 FORMAT (20A4)
      IF (TITLE(1).EQ.CHK) STOP
      WRITE (6,11) TITLE
   11 FORMAT ('1',20A4,//)
C     READ PARAMETERS
      READ (5,15) NEL,NJT,T
```

```
   15 FORMAT (2I10,10X,1G24.16)
      IF (T.GT.0.0) GO TO 1150
      T=1.0D0
 1150 NEQ=NJT*2
      WRITE (6,16) NEL,NJT,T
   16 FORMAT('0','NUMBER OF ELEMENTS IN STRUCTURE',1I10,
     1/,' ','NUMBER OF JOINTS IN STRUCTURE',1I10,
     1/,' ','THICKNESS IS(UNIFORM)=',1G24.16)
C     READ CONNECTIVITY MATRIX
      DO 20 I=1,NEL
   20 READ (5,25) NEID(I), (NCON(I,J),J=1,4)
   25 FORMAT (1I10,4I5)
      WRITE (6,30)
   30 FORMAT('0','CONNECTIVITY MATRIX',//,6X,'EL.ID',4X,'I-J-K-L CONNECT
     1IVITY')
      DO 35 I=1,NEL
   35 WRITE(6,36) NEID(I), (NCON(I,J),J=1,4)
   36 FORMAT (' ',1I10,4I5)
C     READ COORDINATES OF EACH JOINT
      WRITE (6,46)
   46 FORMAT('0',2X,'JOINT ID',12X,'X-COORDINATE',12X,'Y-COORDINATE')
      DO 40 M=1,NJT
      DO 40 N=1,2
   40 COORD(M,N)=0.0D0
      DO 42 M=1,NJT
      READ(5,45) JID(M), (COORD(M,N),N=1,2)
   45 FORMAT(1I10,2G24.16)
   42 WRITE(6,44) JID(M), (COORD(M,N),N=1,2)
   44 FORMAT(' ',1I10,2G24.16)
C     READ FORCES ACTING AT EACH JOINT
      WRITE (6,56)
   56 FORMAT('0',2X,'          ',12X,'HORIZ. FORCE',12X,' VERT. FORCE')
      DO 50 M=1,NEQ
   50 R(M)=0.0D0
      READ (5,55)       (R(M),M=1,NEQ)
   55 FORMAT ( 10X,2G24.16)
   52 WRITE(6,54)      (R(M),M=1,NEQ)
   54 FORMAT (' ', 10X,2G24.16)
C     READ DISPLACEMENTS OF EACH JOINT
      WRITE(6,66)
   66 FORMAT('0',2X,'JOINT ID',10X,'X-DISPLACEMENT',10X,'Y-DISPLACEMENT'
     1)
      DO 60 M=1,NJT
      U(M)=0.0D0
   60 V(M)=0.0D0
      DO 62 M=1,NJT
      READ (5,65) JID(M),U(M),V(M)
   65 FORMAT(1I10,2G24.16)
```

```
   62 WRITE(6,64) JID(M),U(M),V(M)
   64 FORMAT(' ',1I10,2G24.16)
      RETURN
      END

      SUBROUTINE MERGE
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),I0
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN(
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,
     1TITLE(20),T
C     ******************************************************************
C     SUBROUTINE MERGE FORMS THE MASTER MODIFIED STIFFNESS MATRIX
C        BK---MASTER MODIFIED STIFFNESS MATRIX
C        SK---ELEMENT MODIFIED STIFFNESS MATRIX
C        NCON-CONNECTIVITY MATRIX
C        NEL--NUMBER OF ELEMENTS
C        NEQ--NUMBER OF EQUATIONS, NJT*2
C        NJT--NUMBER OF JOINTS IN MASTER STRUCTURE
C        U,V--HORIZ. AND VERT. DISPLACEMENTS AT A JOINT
C        T----THICKNESS (ASSUMED UNIFORM)
C     ******************************************************************
      DO 100 I=1,NEQ
      DO 100 J=1,6
  100 BK(I,J)=0.0D0
C     DETERMINE EACH ELEMENTS MODIFIED STIFFNESS MATRIX
      DO 150 I=1,NJT
      DO 150 J=1,6
      SS(I,J)=0.0D0
  150 SN(I,J)=0.0D0
      DO 250 IO=1,NEL
      CALL STIFF
      DO 200 M=1,4
      IJ=NCON(IO,M)
      II=(M*2)-1
      JJ=II+1
      DO 175 K=1,6
      SS(IJ,K)=SS(IJ,K)+SK(II,K)
      SN(IJ,K)=SN(IJ,K)+SK(JJ,K)
  175 CONTINUE
  200 CONTINUE
  250 CONTINUE
      DO 300 I=1,NJT
      I1=(I*2)-1
      I2=I1+1
      DO 275 K=1,6
      BK(I1,K)=SS(I,K)*T
```

```
      BK(I2,K)=SN(I,K)*T
  275 CONTINUE
  300 CONTINUE
      RETURN
      END


      SUBROUTINE STIFF
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN(
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,
     1TITLE(20),T
      DIMENSION SSK(10)
C     ****************************************************************
C     SUBROUTINE STIFF LOADS THE ELEMENTAL MODIFIED STIFFNESS MATRIX
C        SK---ELEMENT MODIFIED STIFFNESS MATRIX
C        NCON-CONNECTIVITY MATRIX
C        A,B--HORIZ. AND VERT. DIMENSIONS OF AN ELEMENT
C        U,V--HORIZ. AND VERT. DISPLACEMENTS AT A JOINT
C        NEL--NUMBER OF ELEMENTS
C        NJT--NUMBER OF JOINTS IN MASTER STRUCTURE
C     ****************************************************************
      DO 10 I=1,10
   10 SSK(I)=0.0D0
      DO 20 I=1,8
      DO 20 J=1,6
   20 SK(I,J)=0.0D0
C     CALCULATE THE HORIZONTAL DIMENSION, A
      A1=COORD(NCON(IO,1),1)
      A2=COORD(NCON(IO,2),1)
      DA=DABS(A1-A2)
      A1=COORD(NCON(IO,4),1)
      A2=COORD(NCON(IO,3),1)
      DB=DABS(A1-A2)
      A=(DA+DB)/2.0D0
C     CALCULATE THE VERTICAL DIMENSION, B
      A1=COORD(NCON(IO,4),2)
      A2=COORD(NCON(IO,1),2)
      DA=DABS(A1-A2)
      A1=COORD(NCON(IO,3),2)
      A2=COORD(NCON(IO,2),2)
      DB=DABS(A1-A2)
      B=(DA+DB)/2.0D0
C     DEFINE SOME COMMON MULTIPLICATIVE CONSTANTS
      D1=1.0/2.0D0
      D2=1.0/3.0D0
      D3=1.0/4.0D0
```

```
      D4=A/B
      D5=B/A
C     DEFINE THE TERMS OF AUXILIARY ARRAY SSK
      SSK(1)=D1*(U(NCON(IO,1))-U(NCON(IO,3)))
      SSK(2)=D2*D5*(V(NCON(IO,1))-V(NCON(IO,2))-D1*V(NCON(IO,3))+D1*V(NC
     1ON(IO,4)))
      SSK(3)=D1*(-U(NCON(IO,2))+U(NCON(IO,4)))
      SSK(4)=D2*D5*(-D1*V(NCON(IO,1))+D1*V(NCON(IO,2))+V(NCON(IO,3))-V(N
     1CON(IO,4)))
      SSK(5)=D1*(V(NCON(IO,1))-V(NCON(IO,3)))
      SSK(6)=D2*D4*(U(NCON(IO,1))+D1*U(NCON(IO,2))-D1*U(NCON(IO,3))-U(NC
     1ON(IO,4)))
      SSK(7)=D1*(-V(NCON(IO,2))+V(NCON(IO,4)))
      SSK(8)=D2*D4*(D1*U(NCON(IO,1))+U(NCON(IO,2))-U(NCON(IO,3))-D1*U(NC
     1ON(IO,4)))
      SSK(9)=D3*(V(NCON(IO,1))-V(NCON(IO,2))-V(NCON(IO,3))+V(NCON(IO,4))
     1)
      SSK(10)=D3*(U(NCON(IO,1))+U(NCON(IO,2))-U(NCON(IO,3))-U(NCON(IO,4)
     1))
C     LOAD THE ELEMENTAL MODIFIED STIFFNESS MATRIX
      SK(1,1)=D2*D5*(U(NCON(IO,1))-U(NCON(IO,2))-D1*U(NCON(IO,3))+D1*U(N
     1CON(IO,4)))
      SK(1,2)=D3*(V(NCON(IO,1))+V(NCON(IO,2))-V(NCON(IO,3))-V(NCON(IO,4)
     1))
      SK(3,1)=-SK(1,1)
      SK(5,1)=D2*D5*(-D1*U(NCON(IO,1))+D1*U(NCON(IO,2))+U(NCON(IO,3))-U(
     1NCON(IO,4)))
      SK(7,1)=-SK(5,1)
      SK(6,6)=-SSK(10)+SSK(4)
      SK(3,6)=SSK(9)+SSK(8)
      SK(2,6)=SSK(10)+SSK(2)
      SK(1,6)=SSK(9)+SSK(6)
      SK(8,5)=-(SSK(7)+SSK(6))
      SK(6,5)=-(SSK(5)+SSK(8))
      SK(4,5)=SSK(7)+SSK(8)
      SK(2,5)=SSK(5)+SSK(6)
      SK(4,4)=D2*D4*(D1*V(NCON(IO,1))+V(NCON(IO,2))-V(NCON(IO,3))-D1*V(N
     1CON(IO,4)))
      SK(2,4)=D2*D4*(V(NCON(IO,1))+D1*V(NCON(IO,2))-D1*V(NCON(IO,3))-V(N
     1CON(IO,4)))
      SK(7,3)=-(SSK(3)+SSK(4))
      SK(5,3)=-SSK(1)+SSK(4)
      SK(3,3)=SSK(3)-SSK(2)
      SK(1,3)=SSK(1)+SSK(2)
      SK(2,2)=D3*(U(NCON(IO,1))-U(NCON(IO,2))-U(NCON(IO,3))+U(NCON(IO,4)
     1))
      SK(3,2)=-SK(1,2)
      SK(4,2)=SK(2,2)
```

```
      SK(5,2)=-SK(1,2)
      SK(6,2)=-SK(2,2)
      SK(7,2)=SK(1,2)
      SK(8,2)=SK(6,2)
      SK(2,3)=SK(1,1)
      SK(4,3)=-SK(1,1)
      SK(6,3)=SK(5,1)
      SK(8,3)=-SK(5,1)
      SK(6,4)=-SK(4,4)
      SK(8,4)=-SK(2,4)
      SK(1,5)=SK(2,4)
      SK(3,5)=SK(4,4)
      SK(5,5)=SK(6,4)
      SK(7,5)=SK(8,4)
      SK(4,6)=-SK(2,6)
      SK(5,6)=-SK(3,6)
      SK(7,6)=-SK(1,6)
      SK(8,6)=-SK(6,6)
      RETURN
      END


      SUBROUTINE BMULT
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO
C     ******************************************************************
C     SUBROUTINE BMULT PRE-MULTIPLIES THE MASTER STIFFNESS MATRIX AND
C     THE FORCE VECTOR BY THE TRANSPOSE OF BK
C        BK---MASTER MODIFIED STIFFNESS MATRIX
C        BKT--TRANSPOSE OF THE MASTER STIFFNESS MATRIX
C        BSYM-SYMMETRIC SQUARE MATRIX,BKT*BK
C        NEQ--NUMBER OF EQUATIONS,NJT*2
C        R----FORCE VECTOR
C        BR---COLUMN VECTOR,BKT*R
C     ******************************************************************
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN(
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,
     1TITLE(20),T
C     FORM THE TRANSPOSE OF THE MASTER MODIFIED STIFFNESS MATRIX
      DO 10 I=1,6
      DO 10 J=1,NEQ
   10 BKT(I,J)=0.0D0
      DO 20 I=1,6
      DO 20 J=1,NEQ
   20 BKT(I,J)=BK(J,I)
C     FORM THE PRODUCT, BSYM=BKT*BK
      DO 30 I=1,6
      DO 30 J=1,6
```

```
      BSYM(I,J)=0.0D0
      DO 30 K=1,NEQ
   30 BSYM(I,J)=BSYM(I,J)+BKT(I,K)*BK(K,J)
C     FORM THE PRODUCT, BR=BKT*R
      DO 40 I=1,6
      BR(I)=0.0D0
      DO 40 K=1,NEQ
   40 BR(I)=BR(I)+BKT(I,K)*R(K)
      RETURN
      END


      SUBROUTINE SINGL
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN(
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,
     1TITLE(20),T
C     ******************************************************************
C
C     SUBROUTINE SINGL CALCULATES THE EIGEN VALUES ASSOCIATED WITH BSYM.
C     A ZERO EIGEN VALUE INDICATES BSYM TO BE SINGULAR.
C
C     ******************************************************************
      DO 10 I=1,6
      DO 10 J=1,6
   10 ASYM(I,J)=0.0D0
      DO 15 I=1,6
      DO 15 J=1,6
   15 ASYM(I,J)=BSYM(I,J)
      CALL EIGEN (ASYM,6,0,C,A,6)
      WRITE(6,20)
   20 FORMAT('0','EIGEN VALUES OF BSYM=BKT*BK')
      DO 30 I=1,6
   30 WRITE (6,35) C(I)
   35 FORMAT(' ',10X,1G24.16)
      RETURN
      END


      SUBROUTINE EIGEN  (A,N,NOYES,EIVU,EIVR,NDIM)
C.......................................................................
C
C     SUGGESTED FOR USE AT NPG BY PROF. J. G. CANTIN, CODE 59CI
C     ORIGINALLY PROGRAMMED BY W. POOLE, COMPUTER CENTER, U.C.BERKELEY
C
C   PURPOSE
C     THIS SUBROUTINE CALCULATES IN DOUBLE PRECISION ALL THE EIGENVAL-
```

```
C        UES OF A REAL, SYMMETRIC MATRIX BY USE OF THE JACOBI VARIABLE
C        THRESHOLD METHOD.
C
C     USAGE
C        CALL EIGEN (A,N,NOYES,EIVU,EIVR,NDIM)
C
C     DESCRIPTION OF PARAMETERS
C        A   -  THE REAL*8 SYMMETRIC (SQUARE) MATRIX WHOSE EIGENVALUES AND
C               EIGENVECTORS ARE SOUGHT. ROW-DIMENSION MUST BE NDIM.
C        N   -  ORDER OF A, 1.LE.N.LE.NDIM .LE. 160
C        NOYES-INDICATOR SUPPLIED BY USER -
C                      IF NOYES = 0, EIGENVECTORS ARE NOT WANTED,
C                      IF NOYES = 1, EIGENVECTORS ARE WANTED.
C        EIVU - THE EIGENVALUES ARE RETURNED IN THIS VECTOR, WHICH MUST BE
C               DIMENSIONED TO AT LEAST N IN CALLING PROGRAM. THEY ARE IN
C               NO PARTICULAR ORDER. (REAL*8)
         IMPLICIT REAL*8(A-H,O-Z)
C        EIVR - THE EIGENVECTORS (IF REQUESTED) ARE RETURNED AS THE COLUMNS
C               OF THIS MATRIX, THE ROW-DIMENSION OF WHICH MUST BE NDIM IN
C               THE CALLING PROGRAM. THE COLUMN DIMENSION MUST BE AT LEAST
C               N IN THE CALLING PROGRAM. THE VECTOR EIVR(I,J),I=1,N
C               CORRESPONDS TO THE EIGENVALUE EIVU(J). IF EIGENVECTORS
C               ARE NOT REQUESTED, AN UNDIMENSIONED VARIABLE (E.G. DUMMY)
C               SHOULD BE PUT IN THE POSITION OF EIVR IN THE CALL STATE-
C               MENT. EIVR IS REAL*8.
C        NDIM - THE ROW-DIMENSION   (FIRST DIMENSION) OF A AND EIVR IN THE
C               DIMENSION STATEMENT OF THE CALLING PROGRAM. NDIM.LE.160
C
C     REMARKS
C        1)  MATRIX A IS DESTROYED DURING CALCULATION.
C        2)  ALL EIGENVALUES ARE CALCULATED EACH TIME ROUTINE IS CALLED.
C        3)  VARIOUS DIAGNOSTIC MESSAGES MAY BE SENT TO STANDARD OUTPUT
C            MEDIUM BY EIGEN . THESE INDICATE IMPROPER ARGUMENTS AND ARE
C            GENERALLY SELF-EXPLANATORY.
C        4)  LIMIT OF 160 FOR N (AND NDIM) COULD BE RAISED BY CHANGING
C            STATEMENT 21.
C
C     METHOD
C        SEE REFERENCES AND WRITEUP OF F2-BC-JCVT ON FILE AT COMPUTER
C        FACILITY,NPG.
C
         DIMENSION A(NDIM,NDIM),EIVU(NDIM),EIVR(NDIM,NDIM)
C     OBSERVATION
C        TAKEN FROM P.281 OF REFERENCE 2 -
C
C       "EVEN WHEN (THE MATRIX) HAS SOME PATHOLOGICALLY CLOSE EIGENVALUES
C        THE CORRESPONDING EIGENVECTORS ARE ALMOST EXACTLY ORTHOGONAL,
C        THOUGH THEY MAY BE VERY INACCURATE; IT IS EVIDENT THAT THEY SPAN
```

```
C      THE CORRECT SUBSPACE AND GIVE FULL DIGITAL INFORMATION. THIS IS
C      PERHAPS THE MOST SATISFACTORY FEATURE OF JACOBI'S METHOD. THE
C      OTHER TECHNIQUES ... ARE SUPERIOR ... IN SPEED AND ACCURACY, BUT
C      IT IS COMPARATIVELY TIRESOME TO OBTAIN ORTHOGONAL EIGENVECTORS
C      CORRESPONDING TO PATHOLOGICALLY CLOSE OR COINCIDENT EIGENVALUES. "
C
C   REFERENCES
C      1)  NEUMANN, J. VON AND GOLDSTEIN, "THE JACOBI METHOD FOR REAL
C          SYMMETRIC MATRICES", JOURNAL OF ACM, 1, 1959.
C
C      2)  WILKINSON, J.H., "THE ALGEBRAIC EIGENVALUE PROBLEM", OXFORD
C          U. PRESS, 1965, P.277 FF.
C
C.......................................................................
      IF(N-1)20,23,21
   20 PRINT 22,N
   22 FORMAT(4H N= ,I3,68H IS TOO SMALL.  LIMIT IS 1.  RETURN TO CALLING
     1 ROUTINE FROM EIGEN . )
      RETURN
   23 PRINT 24, A(1,1)
   24 FORMAT (24H IN EIGEN   , MATRIX A = ,E14.6)
      RETURN
   21 IF(N-160)1,1,3
    3 PRINT 2,N
    2 FORMAT(3H N=,I5,'   IS TOO LARGE.  LIMIT IS 160.   RETURN TO CALLI
     1NG ROUTINE FROM EIGEN .')
      RETURN
    1 IF(NOYES)99,102,99
   99 CONTINUE
      DO 101 J=1,N
      DO 100 I=1,N
  100 EIVR(I,J)=0.0
  101 EIVR(J,J)=1.0
  102 ATOP=0.
      DO 112 J=1,N
      DO 111 I=1,J
      IF(A(I,J)-A(J,I))90,103,90
   90 CONTINUE
      A(I,J)=.5*(A(I,J)+A(J,I))
      A(J,I)=A(I,J)
  103 CONTINUE
      IF(ATOP-DABS(A(I,J)))104,111,111
  104 ATOP=DABS(A(I,J))
  111 CONTINUE
  112 EIVU(J)=A(J,J)
      IF(ATOP)109,109,113
  109 PRINT 110
  110 FORMAT(26H IN EIGEN , MATRIX A  = 0 )
```

```
      RETURN
  113 AVGF=DFLOAT(N*(N-1))*.55
      D=0.0
      DO 114 JJ=2,N
      DO 114 II=2,JJ
      S=A(II-1,JJ)/ATOP
  114 D=S*S+D
      DSTOP=(1.D-06)*D
      THRSH =DSQRT (D/AVGF)*ATOP
  115 IFLAG=0
      DO 130 JCOL=2,N
      JCOL1=JCOL-1
      DO 130 IROW=1,JCOL1
      AIJ=A(IROW,JCOL)
      IF(DABS(AIJ)-THRSH)130,130,117
  117 AII=A(IROW,IROW)
      AJJ=A(JCOL,JCOL)
      S=AJJ-AII
      IF(DABS(AIJ)-1.D-09*DABS(S))130,130,118
  118 IFLAG=1
      IF(1.D-10*DABS(AIJ)-DABS(S))116,119,119
  119 S=.7071067811865475
      C=S
      GO TO 120
  116 T=AIJ/S
      S=0.25/DSQRT(0.25+T*T)
      C=DSQRT(0.5+S)
      S=2.*T*S/C
  120 DO 121 I=1,IROW
      T=A(I,IROW)
      U=A(I,JCOL)
      A(I,IROW)=C*T-S*U
  121 A(I,JCOL)=S*T+C*U
      I2=IROW+2
      IF(I2-JCOL)127,127,123
  127 CONTINUE
      DO 122 I=I2,JCOL
      T=A(I-1,JCOL)
      U=A(IROW,I-1)
      A(I-1,JCOL)=S*U+C*T
  122 A(IROW,I-1)=C*U-S*T
  123 A(JCOL,JCOL)=S*AIJ+C*AJJ
      A(IROW,IROW)=C*A(IROW,IROW)-S*(C*AIJ-S*AJJ)
      DO 124 J=JCOL,N
      T=A(IROW,J)
      U=A(JCOL,J)
      A(IROW,J)=C*T-S*U
  124 A(JCOL,J)=S*T+C*U
```

```
      IF(NOYES)131,126,131
  131 CONTINUE
      DO 125 I=1,N
      T=EIVR(I,IROW)
      EIVR(I,IROW)=C*T-EIVR(I,JCOL)*S
  125 EIVR(I,JCOL)=S*T+EIVR(I,JCOL)*C
  126 CONTINUE
      S=AIJ/ATOP
      D=D-S*S
      IF(D-DSTOP)1260,129,129
 1260 D=0.
      DO 128 JJ=2,N
      DO 128 II=2,JJ
      S=A(II-1,JJ)/ATOP
  128 D=S*S+D
      DSTOP=(1.D-06)*D
  129 THRSH=DSQRT(D/AVGF)*ATOP
  130 CONTINUE
      IF(IFLAG)115,134,115
  134 T=A(1,1)
      A(1,1)=EIVU(1)
      EIVU(1)=T
      DO 132 J=2,N
      T=A(J,J)
      A(J,J)=EIVU(J)
      EIVU(J)=T
      DO 132 I=2,J
  132 A(I-1,J)=A(J,I-1)
  133 RETURN
      END
C
C     ..................................................................
C
C        SUBROUTINE INVRT
C
C        PURPOSE
C           INVERT A MATRIX
C
C        USAGE
C           CALL INVRT(A,N,D,L,M)
C
C        DESCRIPTION OF PARAMETERS
C           A AND D MUST BE REAL*8
C           A - INPJT MATRIX, DESTROYED IN COMPUTATION AND REPLACED BY
C               RESULTANT INVERSE.
C           N - ORDER OF MATRIX A
C           D - RESULTANT DETERMINANT
C           L - WCRK VECTOR OF LENGTH N
```

```
C          M - WORK VECTOR OF LENGTH N
C
C       REMARKS
C          MATRIX A MUST BE A GENERAL MATRIX
C
C       SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C          NONE
C
C       METHOD
C          THE STANDARD GAUSS-JORDAN METHOD IS USED. THE DETERMINANT
C          IS ALSO CALCULATED. A DETERMINANT OF ZERO INDICATES THAT
C          THE MATRIX IS SINGULAR.
C
C     DOCUMENTATION:
C                  SUBROUTINE INVRT IS A COPY OF SUBROUTINE DMINV,
C                  A CATALOGUED LIBRARY ROUTINE FROM THE W.R.CHURCH
C                  COMPUTER CENTER, NAVPGSCOL, MONTEREY, CALIF. 93940
C     ...............................................................
C

      SUBROUTINE INVRT(A,N,D,L,M)
      DIMENSION A(1),L(1),M(1)
      DOUBLE PRECISION A,D,BIGA,HOLD
C
C        .............................................................
C
C        SEARCH FOR LARGEST ELEMENT
C
      D=1.0D0
      NK=-N
      DO 80 K=1,N
      NK=NK+N
      L(K)=K
      M(K)=K
      KK=NK+K
      BIGA=A(KK)
      DO 20 J=K,N
      IZ=N*(J-1)
      DO 20 I=K,N
      IJ=IZ+I
   10 IF(DABS(BIGA)-DABS(A(IJ))) 15,20,20
   15 BIGA=A(IJ)
      L(K)=I
      M(K)=J
   20 CONTINUE
C
C        INTERCHANGE ROWS
```

```
C
      J=L(K)
      IF(J-K) 35,35,25
   25 KI=K-N
      DO 30 I=1,N
      KI=KI+N
      HOLD=-A(KI)
      JI=KI-K+J
      A(KI)=A(JI)
   30 A(JI) =HOLD
C
C        INTERCHANGE COLUMNS
C
   35 I=M(K)
      IF(I-K) 45,45,38
   38 JP=N*(I-1)
      DO 40 J=1,N
      JK=NK+J
      JI=JP+J
      HOLD=-A(JK)
      A(JK)=A(JI)
   40 A(JI) =HOLD
C
C        DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS
C        CONTAINED IN BIGA)
C
   45 IF(BIGA) 48,46,48
   46 D=0.0D0
      RETURN
   48 DO 55 I=1,N
      IF(I-K) 50,55,50
   50 IK=NK+I
      A(IK)=A(IK)/(-BIGA)
   55 CONTINUE
C
C        REDUCE MATRIX
C
      DO 65 I=1,N
      IK=NK+I
      HOLD=A(IK)
      IJ=I-N
      DO 65 J=1,N
      IJ=IJ+N
      IF(I-K) 60,65,60
   60 IF(J-K) 62,65,62
   62 KJ=IJ-I+K
      A(IJ)=HOLD*A(KJ)+A(IJ)
   65 CONTINUE
```

```
C
C         DIVIDE ROW BY PIVOT
C
      KJ=K-N
      DO 75  J=1,N
      KJ=KJ+N
      IF(J-K)  70,75,70
   70 A(KJ)=A(KJ)/BIGA
   75 CONTINUE
C
C         PRODUCT OF PIVOTS
C
      D=D*BIGA
C
C         REPLACE PIVOT BY RECIPROCAL
C
      A(KK)=1.0/BIGA
   80 CONTINUE
C
C         FINAL ROW AND COLUMN INTERCHANGE
C
      K=N
  100 K=(K-1)
      IF(K)  150,150,105
  105 I=L(K)
      IF(I-K)  120,120,108
  108 JQ=N*(K-1)
      JR=N*(I-1)
      DO 110  J=1,N
      JK=JQ+J
      HOLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
  110 A(JI) =HOLD
  120 J=M(K)
      IF(J-K)  100,100,125
  125 KI=K-N
      DO 130  I=1,N
      KI=KI+N
      HOLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
  130 A(JI) =HOLD
      GO TO 100
  150 RETURN
      END
```

```
      SUBROUTINE ANSER
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/INT/NCON(50,4),L1(6),L2(6),NEL,NJT,NEQ,NEID(50),JID(100),IO
      COMMON/FLP/BK(200,6),BKT(6,200),R(200),COORD(100,2),SS(100,6),SN(
     1100,6),SK(8,6),BSYM(6,6),ASYM(6,6),U(100),V(100),BR(6),C(6),A,B,
     1TITLE(20),T
C     ******************************************************************
C
C     SUBROUTINE ANSER OUTPUTS THE SIX MATERIAL CONSTANTS.  THE VECTOR
C     OF MATERIAL CONSTANTS IS GENERATED BY MULTIPLICATION OF THE VECTOR
C     BR BY THE INVERSE OF BSYM.
C
C     ******************************************************************
      DO 10 J=1,6
   10 C(J)=0.0D0
      DO 20 K=1,6
      DO 20 L=1,6
   20 C(K)=C(K)+BSYM(K,L)*BR(L)
      WRITE(6,50)
   50 FORMAT('1','THE VECTOR OF MATERIAL CONSTANTS',
     1//,' TRANSPOSE<C11,C12,C13,C22,C23,C33>=')
      DO 60 I=1,6
   60 WRITE(6,70) C(I)
   70 FORMAT('0',15X,1G24.16)
      RETURN
      END
```

REPRESENTATIVE OUTPUT: 15 KIP LOAD, 4-NODES. (DLASTIC)

```
NUMBER OF ELEMENTS IN STRUCTURE          4
NUMBER OF JOINTS IN STRUCTURE          9
THICKNESS IS(UNIFORM)=  0.10000000000000000D 00

CONNECTIVITY MATRIX

     EL.ID     I-J-K-L  CONNECTIVITY
       1       2    3    6    5
       2       5    6    9    8
       3       1    2    5    4
       4       4    5    8    7

   JOINT ID            X-COORDINATE                 Y-COORDINATE
       1     10.0000000000000000      0.0
       2      5.0000000000000000      0.0
       3    0.0                       0.0
       4     10.0000000000000000    -5.0000000000000000
       5      5.0000000000000000    -5.0000000000000000
       6    0.0                     -5.0000000000000000
       7     10.0000000000000000    -10.0000000000000000
       8      5.0000000000000000    -10.0000000000000000
       9    0.0                     -10.0000000000000000

                       HORIZ. FORCE                     VERT. FORCE
            0.91039551760226l9        3.9241564875 58248
            0.0                        7.1516870248 83492
           -0.9103955176022620        3.9241564875 58250
            0.0                        0.0
            0.0                        0.0
            0.0                        0.0
            0.0                       -5.0000000000000000
            0.0                       -5.0000000000000000
            0.0                       -5.0000000000000000

   JOINT ID          X-DISPLACEMENT                 Y-DISPLACEMENT
       1    0.0                       0.0
       2    0.0                       0.0
       3    0.0                       0.0
       4   -0.4929293242488149D-03  -0.2617793655895807D-02
       5    0.2014108968973132D-18  -0.2189597399146653D-02
       6    0.4929293242488153D-03  -0.2617793655895808D-02
       7   -0.1315568119229649D-02  -0.5164481934212524D-02
       8    0.4872528994009446D-18  -0.3827868226091310D-02
       9    0.1315568119229649D-02  -0.6164481934212524D-02

EIGEN VALUES OF BSYM=BKT*BK
            0.2031967468551705D-08
            0.2443927086398269D-06
            0.3938446247547611D-07
            0.1542630641666066D-06
            0.3839002888303476D-06
            0.3825140881455582D-07

THE VECTOR OF MATERIAL CONSTANTS

TRANSPOSE<C11,C12,C13,C22,C23,C33>=

                    32147.99848740955

                    9226.475565886535

                  -0.5826520139618010D-11

                    32147.99848740942

                  -0.3403485991231466D-11

                    11460.76146076146
```

BIBLIOGRAPHY

1. Abramowitz, M. and I. A. Segun (eds.), "2500 Five Digit Random Numbers", in Handbook of Mathematical Functions, Dover, New York, 1968.

2. Bychawski, Z. and A. Fox, "The Constitutive Equation of a Complete Nonlinear Viscoelastic Material", Acta Mechanica, 6, 2-3, 1968.

3. Cantin, G., Naval Postgraduate School, Monterey, California, private communication (1972).

4. Clough, R. W., "The Finite Element Method in Structural Mechanics", in Stress Analysis, O. C. Zienkiewicz and G. S. Holister (eds), Wiley, London, 1965.

5. Desai, C. S. and J. F. Abel, Introduction to the Finite Element Method, Van Nostrand-Reinhold, New York, 1972.

6. Dong, R., K. Pister, and R. Dunham, "Mechanical Characterization of Non-linear Viscoelastic Solids for Iterative Solutions of Boundary Value Problems", Acta Mechanica, 9, 1-2, 1970.

7. Evans, R. J. and K. Pister, "Constitutive Equations for a Class of Non-linear Elastic Solids", International Journal of Solids and Structures, 2, 4, 1966.

8. Filonenko-Borodich, M. M., Theory of Elasticity, (M. Konayeva, translator), Noordhoff, Groningen (The Netherlands), undated.

9. Gleason, R. F., "On the Determination of Material Characteristics in Nonlinear Viscoelasticity", (U), U. S. Air Force Rocket Propulsion Laboratory, Stanford University, AFRPL-TR-67-192(1967).

10. Hearmon, R. F. S., An Introduction to Applied Anisotropic Elasticity, Oxford University, Oxford, 1961.

11. Jaensson, B. O. and B. O. Sundström, "Determination of Young's Modulus and Poisson's Ratio for WC-Co Alloys by the Finite Element Method", Materials Science and Engineering, 9, 4, 1972.

12. Kavanagh, K. T., The Finite Element Analysis of Physically and Kinematically Nonlinear Elastic Solids, Ph.D. dissertation, University of California (Berkeley), Berkeley, California (1969).

13. Kavanagh, K. T., and R. W. Clough, "Finite Element Applications in the Characterization of Elastic Solids", International Journal of Solids and Structures, 7, 1971.

14. Muskhelishvili, N. I., Some Basic Problems of the Mathematical Theory of Elasticity, (J. R. M. Radok, translator), Noordhoff, Groningen (The Netherlands), 1953.

15. Nachlinger, R. R. and H. H. Calvit, "On Approximate Constitutive Equations in Nonlinear Viscoelasticity", Acta Mechanica, 7, 4, 1969.

16. Oden, J. T., "Numerical Formulations of Non-linear Elasticity Problems", Journal of the American Society of Civil Engineers (Structural), 93, 3, 1967.

17. Oden, J. T., "A General Theory of Finite Elements", International Journal of Numerical Methods in Engineering, 1, 1969.

18. Sandhu, R. S. and K. S. Pister, "Variational Principles for Boundary Value and Initial-boundary Value Problems in Continuum Mechanics", International Journal of Solids and Structures, 7, 1971.

19. Timoshenko, S. and J. N. Goodier, Theory of Elasticity, McGraw-Hill, New York, 1951.

20. Washizu, K., Variational Methods in Elasticity and Plasticity, Pergamon, Oxford, 1968.

21. Zienkiewicz, O. C., The Finite Element Method in Engineering Science, McGraw-Hill, London, 1971.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Documentation Center Cameron Station Alexandria, Virginia 22314	2
2.	Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
3.	Professor Gilles Cantin, Code 59Ci Department of Mechanical Engineering Naval Postgraduate School Monterey, California 93940	5
4.	Asst Professor David Salinas, Code 59Zc Department of Mechanical Engineering Naval Postgraduate School Monterey, California 93940	1
5.	Dr. A. Adicoff Code 6056 Naval Weapons Center China Lake, California 93955	1
6.	Professor Ray Clough Department of Civil Engineering Division of Structural Mechanics University of California Berkeley, California 94720	1
7.	LT Brian A. Edwards USN 2 Baxter Avenue New Hyde Park, New York 11040	1

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

REPORT TITLE

Characterization of Elastic Solids Using Finite Element Methods

DESCRIPTIVE NOTES (Type of report and, inclusive dates)

Master's Thesis; December 1972

AUTHOR(S) (First name, middle initial, last name)

Brian A. Edwards

REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1972	94	21

CONTRACT OR GRANT NO. PROJECT NO.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Naval Postgraduate School Monterey, California 93940

ABSTRACT

Finite element methods are applied to the problem of characterizing linear, anisotropic elastic solids. The conventional finite element displacement formulation is used to simulate an elastic material in plane stress. An inverted finite element formulation is then applied, and the characterizing six material constants are calculated as numerical results.

A possible test device for the experimental characterization of anisotropic solids is postulated, the precision of displacement measurements to be required for such a device being determined by random perturbation analysis. Numerical constants accurate to within three percent are predicted if a precision of one part in eight hundred (1/800) can be measured. Numerical constants accurate to within one percent are predicted if a precision of one part in eight thousand (1/8000) can be measured in the test device.

DD FORM 1473 (PAGE 1)
1 NOV 65

S/N 0101-807-6811

UNCLASSIFIED
Security Classification

A-31408

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
CHARACTERIZATION						
MATERIAL CONSTANT						
FINITE ELEMENT						
ANISOTROPIC						
ELASTICITY						
ELASTIC SOLID						